

Field Theoretical Methods in Cosmology

by

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Abstract

To optimally utilize all the exciting cosmological data coming in we need to sharpen also the theoretical tools available to cosmologists. One such indispensable tool to understand hot big bang cosmology is finite temperature field theory. We review and summarise the efforts made by us to use finite temperature field theory to address issues of current interest to cosmologists. An introduction to both the real time and the imaginary time formalisms is provided.

The imaginary time formalism is illustrated by applying it to understand the interesting possibility of Late Time Phase Transitions. Recent observations of the space distribution of quasars indicate a very notable peak in space density at a redshift of 2 to 3. It is pointed out that this may be the result of a phase transition which has a critical temperature of roughly a few meV (in the cosmological units, $\hbar = c = k = 1$), which is natural in the context of massive neutrinos. In fact, the neutrino masses required for quasar production and those required to solve the solar neutrino problem by the MSW mechanism are consistent with each other. As a bonus, the cosmological constant implied by this model may also help resolve the discrepancy between the recently measured value of the Hubble Constant and the age of the universe.

We illustrate the real time formalism by studying one of the most important time-dependent and non-equilibrium phenomena associated with phase transitions. The non-equilibrium dynamics of the first stage of the reheating process, that is dissipation via particle production is studied in scalar field theories. We show that a complete understanding of the mechanism of dissipation via particle production requires a non-perturbative resummation. We then study a Hartree approximation and clearly exhibit dissipative effects related to particle production. The effect of dissipation by Goldstone bosons is studied non-

perturbatively in the large N limit in an $O(N)$ theory.

We also place our work in perspective and point out some of the related issues which clearly need further exploration.

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Chapter 1

INTRODUCTION TO FINITE TEMPERATURE FIELD THEORY AND ITS RELEVANCE TO COSMOLOGY

1.1 Introduction

Cosmology is one of the most rapidly growing fields of physics today. This is so because there is a fairly large number of observations which we are now in position to make which have been impossible to do so far in mankind's history. The combination of both space based observations and improvements in technology and techniques of ground based observations is dramatically improving both the quantity and quality of data now available. These data will clearly lead to an improved understanding of the physics of the universe we live in.

There are many examples of areas of cosmology where better data continues to lead to better understanding. In particular, structure formation continues to benefit from data from a variety of sources. These data includes improving determination

of the clustering of galaxies and other larger scale structures[13]. Further light is being shed on structure formation by examining number densities of quasars[1], clustering properties of quasars[3] and the study of Lyman- α clouds[4]. A particularly important piece of information that has direct implications for structure formation has become available in recent years - the detection of anisotropies in the Cosmic Microwave Background Radiation(CMBR)[12]. This has been measured by COBE and continues to be pursued by ground based and balloon based experiments. We are starting to get information on the CMBR anisotropy on a variety of angular scales and this together with available data on the clustering of galaxies and other larger scale structures is starting to severely constrain models of structure formation.

Another example of fresh observations which have the potential of leading to an improved understanding of the physics of the universe we live in is the accurate determination of the Hubble Constant using Cepheid variables as standard candles[5, 6]. This is among the many achievements of the new refurbished Hubble Space Telescope. The ability of the Space Telescope to see at a level of detail which has not been possible earlier continues to have an impact on many fields of astrophysics. Another of its significant contribution has been to provide compelling evidence for the existence of supermassive blackholes at the center of the galaxies. An examination of the rotation velocities of objects close to the center of the galaxy implied the existence of an enormously massive object confined to such a small region of space that there is no known mechanism of preventing the collapse of such a massive object to form a black hole.

The determination of the Hubble Constant is an example where both previously impossible space based observations[6] and improvements in ground based

techniques and technology[5] are happening at the same time to complement and confirm improved, more accurate results.

The accurate determination of the Hubble constant is also an example of a recent observation that has important implications for cosmology and the potential of leading to an improved understanding of the physics of the universe we live in. One can use the Hubble constant to determine the “expansion age” of the universe which is the age of the universe derived on the assumption that we live in a flat universe with no vacuum energy. This expansion age may turn out to be less than the observed ages of globular star clusters. If all the data stand up to the test of time then clearly we have a very important piece of fresh information about the universe.

One of the ways of resolving this apparent conflict between the age of the universe and the Hubble constant is to introduce a cosmological constant in Einstein’s equations which determine the evolution of the universe. The naturalness or unnaturalness of this solution is discussed in a later chapter and will be commented on later, but there is a further point about the role of observations in this context that we would like to make here.

The existence and magnitude of the cosmological constant is an issue which is also benefitting from continued improvements in astronomical observations. In particular, since the cosmological constant modifies the geometry of the universe and so influences the propagation of light it can be observationally determined or at least constrained. Thus by observing such quantities as quasar lensing frequencies and comparing them to those expected for various values of the cosmological constant one can determine or at least constrain the value of the cosmological constant[7]. There is admittedly a dependence on the modelling of the time evolu-

tion due to ongoing structure formation involved, but for a given detailed model of structure formation one can use the observations to shed light on the cosmological constant issue. Indeed, considerable progress has been made in this field too in the last few years and continued improvements will only make this a further source of rich information on cosmology.

So far we have given examples of ongoing observations that are yielding important information about cosmology. There is also a host of observations that will be coming on-line in the future which will undoubtedly give us valuable new information about cosmology. In particular, neutrino astronomy[8] and gravitational wave astronomy[9] will allow us to take a look at the universe in a way we have never been able to explore previously.

It is clear that cosmology is becoming an intensely data-driven subject. The fresh new data that continues to stream in will undoubtedly lead to an increased understanding of the universe we live in. However, to use all the data efficiently and effectively, utilize it to yield insights into the fundamental physics of the processes involved requires us also to equip ourselves with some theoretical tools and techniques.

Field theory is clearly one of the essential tools for an understanding of cosmology. In particular, finite temperature field theory plays a central role in understanding hot big bang cosmology. This is perhaps underlined by the recent COBE observation of the perfect thermal Planckian spectrum for the photon gas that inhabits the universe. The Hubble expansion then tells us that at earlier times the energy density was higher leading to higher temperatures for the photon gas at earlier epochs. Also at the earlier higher densities the higher interaction rates would imply that there was thermal equilibration of the various particle distributions

with the photon gas at earlier epochs.

In addition to this we believe there were a number of phase transitions during the evolution of the universe. These phase transitions have important implications for cosmology. One of these phase transitions is believed to be responsible for inflation which solves many of the problems of standard cosmology, such as the flatness problem and the entropy problem[1, 2]. The natural tool to study all these phenomena is finite temperature field theory. This point has been made in a number of excellent contemporary books on cosmology[4, 5].

In this thesis, we attempt to provide an introduction to finite temperature field theory geared towards those interested in cosmology. Towards this goal we have used the techniques of finite temperature field theory to address some of the subjects and issues of current interest to cosmologists.

Phase transitions play an important role in modern theories of particle physics and cosmology[4, 5]. Symmetry breaking phase transitions are an integral part of our current understanding of the unification of fundamental interactions[39].

In the Big Bang cosmology as the universe evolves it expands and it cools. Starting at high temperatures with a very symmetric universe as the universe expands and cools it goes through a sequence of phase transitions. Thus, first the GUT symmetry occurs followed by the electroweak symmetry breaking followed by other phase transitions.

Among the interesting cosmological issues that are intimately linked with these phase transitions are the formation and evolution of topological defects[40, 41, 43, 42, 44, 16], inflation[46, 48, 47, 49], structure formation and baryogenesis.

It is clear that we need a systematic framework in which to address all of these issues. Finite temperature field theory[50, 51, 52, 53, 55, 56] is definitely the

framework one needs to use to get detailed quantitative information about these interesting and important events. Within the framework of finite temperature field theory there are two different formalisms one could use. These are respectively the imaginary time formalism and the real time formalism of finite temperature field theory. Depending on the particular questions or issues one wishes to address one formalism may be better than the other and on occasion one of the formalisms may not be relevant to the issues to be addressed. In particular, to determine the existence of phase transitions, the magnitude of the critical temperature and the nature of topological defects formed at the phase transition one may wish to use the imaginary time formalism. However to study the time dependent and non-equilibrium phenomena associated with phase transitions the natural tool to use is the real time formalism of finite temperature field theory. The various uses of the two different formalisms of finite temperature field theory will be illustrated in detail by studying specific examples of current interest.

Phase transitions that occur after the decoupling of matter and radiation have been discussed in the literature as Late Time Phase Transitions (LTPT's). The original motivation for considering LTPT's[11] [16] [17] [25] [20] [23] was the need to reconcile the extreme isotropy of the Cosmic Microwave Background Radiation (CMBR)[19] with the existence of large scale structure[13] and also the existence of quasars at high redshifts[14].

Discussions of realistic particle physics models capable of generating LTPT's have been carried out by several authors[20] [25]. It has been pointed out that the most natural class of models in which to realise the idea of LTPT's are models of neutrino masses with Pseudo Nambu Goldstone Bosons (PNGB's). The reason for this is that the mass scales associated with such models can be related to the

neutrino masses, while any tuning that needs to be done is protected from radiative corrections by the symmetry that gave rise to the Nambu-Goldstone modes[21].

Holman and Singh[23] studied the finite temperature behaviour of the see-saw model of neutrino masses and found phase transitions in this model which result in the formation of topological defects. In fact, the critical temperature in this model is naturally linked to the neutrino masses.

The original motivation for studying the finite temperature behaviour of the see-saw model of neutrino masses came from a desire to find realistic particle physics models for Late Time Phase Transitions. It now appears that this may also provide a physically appealing and observationally desirable magnitude for the cosmological constant.

In particle physics one of the standard ways of generating neutrino masses has been the see-saw mechanism [22]. These models involve leptons and Higgs fields interacting by a Yukawa type interaction. We computed the finite temperature effective potential of the Higgs fields in this model. An examination of the manifold of degenerate vacua at different temperatures allowed us to describe the phase transition and the nature of the topological defects formed.

To investigate in detail the finite temperature behaviour of the see-saw model we selected a very specific and extremely simplified version of the general see-saw model. However, we expect some of the qualitative features displayed by our specific simplified model to be at least as rich as those present in more complicated models.

In the following chapter we will study in some detail a specific simple version of the see-saw model of neutrino masses[22]. By studying the finite temperature behavior of this model Holman and Singh[23] demonstrated the existence of phase

transitions, establishing the magnitude of the critical temperature. This was done by using the imaginary time formalism of finite temperature field theory. By studying the manifold of degenerate vacua we also established the nature of topological defects that would be formed. The phase transition implied by this model may have interesting cosmological consequences if we use the neutrino masses implied by the MSW solution[29] to the solar neutrino problem. In particular, these phase transitions might help explain the peak in the quasar space densities observed at redshifts between 2 and 3[16]. Also, the magnitude of the cosmological constant implied by this model might help resolve the apparent conflict between the age of the universe and the Hubble constant[41]. We will study the details of the finite temperature see-saw model in Chapter 2.

The real time formalism of finite temperature field theory[58, 59, 56] has been used by us to study time dependent and non-equilibrium phenomena during and following a phase transition such as domain formation and growth[16], thermal activation[60], dissipation via particle production and reheating[61, 62]. In Chapter 3 we will study in detail the mechanism of dissipation via particle production as an illustration of the use of the real time formalism to study time dependent phenomena following a phase transition.

We next review the formalism we will use in later chapters. The remainder of this chapter will be devoted to introducing first the imaginary and then the real time formalisms of finite temperature field theory.

1.2 The Imaginary Time Formalism

In finite temperature field theory the basic quantity of interest are the finite temperature Green's functions. To study phase transitions and identify the critical temperature and nature of topological defects one uses the effective potential which can be expressed in terms of 1 particle irreducible Greens functions as discussed below.

Let us quickly review the zero temperature field theory and the zero temperature effective potential first and then discuss their finite temperature counterparts. We will consider the simplest possible example of a scalar field theory to introduce the formalism[39, 54] .

The starting point of our discussion is the generating functional for the Greens functions,

$$W[J] = \int D\phi \exp \left\{ i \int d^4x (L + J\phi) \right\} \quad (1.1)$$

where L is the Lagrangian density describing the field theory of interest, ϕ is the scalar field of interest and J is the source which allows us to express the n-point greens functions in terms of the generating functional in the following way:

$$G^{(n)}(x_1, x_2, \dots, x_n) = \frac{1}{i^n} \frac{1}{W} \left[\frac{\delta^n W[J]}{\delta J(x_1) \dots \delta J(x_n)} \right]_{J=0} . \quad (1.2)$$

Similarly we can introduce the quantity, $Z[J] \equiv -i \ln W[J]$, which can be shown to be the generating functional of connected greens functions[39],

$$G_c^{(n)}(x_1, x_2, \dots, x_n) = \left[\frac{\delta^n \ln W[J]}{\delta J(x_1) \dots \delta J(x_n)} \right]_{J=0} . \quad (1.3)$$

Then, the classical field which is the vacuum expectation value of the field is given by,

$$\phi_c(x) \equiv \frac{\langle 0|\phi(x)|0\rangle}{\langle 0|0\rangle} = \frac{\delta Z[J]}{\delta J(x)}. \quad (1.4)$$

The effective action, Γ is the functional Legendre transform of the generating functional $Z[J]$,

$$\Gamma[\phi_c] = Z[J] - \int d^4x J(x)\phi_c(x). \quad (1.5)$$

One can also expand $\Gamma[\phi_c]$ in powers of ϕ_c as,

$$\Gamma[\phi_c] = \sum_n \frac{1}{n!} \int d^4x_1 \dots d^4x_n \Gamma^{(n)}(x_1, x_2, \dots x_n) \phi_c(x_1) \dots \phi_c(x_n). \quad (1.6)$$

It can be shown that, $\Gamma^{(n)}(x_1, x_2, \dots x_n) \equiv \frac{\delta^n \Gamma}{\delta \phi_c(x_1) \dots \delta \phi_c(x_n)}$ are the 1 particle irreducible (1PI) n-point greens functions. Recall that the 1PI Greens functions receive contributions only from Feynman diagrams which cannot be disconnected by cutting any one internal line.

Making a derivative expansion of the effective action allows us to isolate the effective potential as the lowest order term in the expansion,

$$\Gamma[\phi_c] = \int d^4x \left[-V_{eff}(\phi_c) + \frac{1}{2}(\partial_\mu \phi_c)^2 F(\phi_c) + \dots \right]. \quad (1.7)$$

In particular, for $\phi_c(x) = \phi_c = \text{constant}$, we have $\partial_\mu \phi_c = 0$. Thus, by taking Fourier transforms one obtains from the previous equations the following relation,

$$V_{eff}(\phi_c) = - \sum_n \frac{1}{n!} \Gamma^{(n)}(p_1 = 0, \dots p_n = 0) \phi_c^n. \quad (1.8)$$

1.2.1 Calculation of the effective potential using the tad-pole method

The effective potential formalism is the most well-known and widely used formalism to study the phenomenon of spontaneous symmetry breaking in field theory. The formalism is by now well-developed and there exist a number of excellent discussions on the effective potential and methods to calculate it[54, 39]. The most straightforward and frequently discussed method of calculating the effective potential is by evaluating all the 1PI n-point Greens function and then explicitly performing the sum over n as required by the relation given earlier to obtain an expression for the effective potential. There exists another method which is less well-known, a little devious, usually easier to implement in practice and is our method of choice to calculate the effective potential. By being a little devious we can express the effective potential completely in terms of a 1-point 1PI Greens functions and this makes the computaion of effective potential much simpler in practice. We now turn to a description of this method and its illustration using a simple example.

By introducing a shift in the fields,

$$\phi \rightarrow \phi - \omega \tag{1.9}$$

one can obtain an expression for the effective potential in terms of the 1PI n-point functions in the shifted theory, $\Gamma_{\omega}^{(n)}$ which is more useful for actually calculating the effective potential in practice. Thus,

$$V_{eff}(\phi_c) = - \sum_n \frac{1}{n!} \Gamma_{\omega}^{(n)}(p_1 = 0, \dots p_n = 0) (\phi_c - \omega)^n. \tag{1.10}$$

This implies,

$$\left[\frac{dV_{eff}}{d\phi} \right]_{\phi=\omega} = -\Gamma_{\omega}^{(1)}(p_1 = 0). \quad (1.11)$$

Therefore, we have,

$$V_{eff} = - \int \Gamma_{\omega}^{(1)} d\omega. \quad (1.12)$$

Note that by making the above transformations we can calculate the effective potential by knowing only the 1-point 1 PI function in the shifted theory as opposed to calculating all the n-point functions and then summing over n in the original expression for the effective potential. This method is called the tadpole method because the calculation of the 1PI 1-point function, $\Gamma_{\omega}^{(1)}$ involves the computation of a Feynman diagram which bears a closer resemblance to a tadpole than the penguin diagram's resemblance to a penguin.

Let us illustrate the method with a simple example. Consider a scalar field theory with the following tree level potential,

$$V_0(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4. \quad (1.13)$$

Performing the shift, $\phi \rightarrow \phi - \omega$, we get,

$$V_{\omega} = \frac{1}{2}m^2(\phi - \omega)^2 + \frac{\lambda}{4!}(\phi - \omega)^4. \quad (1.14)$$

In the shifted theory the mass of the ϕ field, determined from the coefficient of the ϕ^2 term is given by, $M_{\phi}^2(\omega) \equiv m^2 + \frac{\lambda\omega^2}{2}$. The computation of $\Gamma_{\omega}^{(1)}$ involves the cubic self-coupling i.e. the coefficient of the ϕ^3 term which is $\frac{\lambda}{3!}\omega$. Thus, the 1 loop contribution to $\Gamma_{\omega}^{(1)}$ is,

$$\begin{aligned}
\Gamma_{\omega}^{(1)}(p_1 = 0) &= -\frac{i}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{\lambda \omega}{k^2 - M_{\phi}^2(\omega) + i\epsilon} \\
&= -\frac{dV_{eff}}{d\omega}.
\end{aligned} \tag{1.15}$$

Thus,

$$\begin{aligned}
V_{eff} &= - \int \Gamma_{\omega}^{(1)} d\omega \\
&= \frac{i}{2} \int \frac{d^4 k}{(2\pi)^4} \ln \left[\frac{k^2 - M_{\phi}^2(\omega) + i\epsilon}{\Lambda^2} \right].
\end{aligned} \tag{1.16}$$

By performing the rotation to Euclidean space the above integral can be evaluated to yield the 1 loop contribution to the effective potential for this theory to be,

$$V_{eff} = \frac{1}{64\pi^2} \left\{ -\frac{\Lambda^4}{2} + 2\Lambda^2 M_{\phi}^2(\omega = \phi_c) + M_{\phi}^4(\omega = \phi_c) \left[\ln \left(\frac{M_{\phi}^2(\omega = \phi_c)}{\Lambda^2} \right) - \frac{1}{2} \right] \right\} \tag{1.17}$$

where Λ is the momentum cutoff introduced to regularise the integral. This theory can now be renormalized by adding counterterms and specifying a renormalization prescription as is done below. The counterterms we add are

$$\delta V_{ct} = -\frac{1}{2} \delta m^2 \phi_c^2 - \frac{\delta \lambda}{4!} \phi_c^4 + \delta V. \tag{1.18}$$

The total potential, V then becomes the sum of the bare potential and δV_{ct} . We specify the following renormalization prescription:

$$\left[\frac{d^2 V}{d\phi^2} \right]_{\phi=0} = m_r^2, \quad \left[\frac{d^4 V}{d\phi^4} \right]_{\phi=0} = \lambda_r, \quad V(\phi = 0) = V_o. \tag{1.19}$$

This yields the following expressions for the renormalized parameters:

$$V_o = \delta V + \frac{m^4}{32\pi^2} \left[\ln \left(\frac{m^2}{\Lambda^2} \right) - \frac{1}{2} \right] \quad (1.20)$$

$$m_r^2 = \delta m^2 + \frac{\lambda \Lambda}{32\pi^2} + m^2 + \frac{\lambda m^2}{32\pi^2} \ln \frac{m^2}{\Lambda^2} \quad (1.21)$$

$$\lambda_r = \lambda + \delta \lambda \frac{3\lambda^2}{32\pi^2} \left[\ln \left(\frac{m^2}{\Lambda^2} \right) + 1 \right]. \quad (1.22)$$

$$(1.23)$$

Thus in terms of these renormalized parameters the complete effective potential becomes,

$$V_{renorm} = \frac{1}{2} m_r^2 \phi^2 + \frac{\lambda_r}{4!} \phi^4 + \frac{m_\phi^4}{64\pi^2} \left[\ln \left(\frac{m_\phi^2}{\Lambda^2} \right) - \frac{1}{2} \right] + V_o \quad (1.24)$$

where $m_\phi^2 = m_r^2 + \frac{\lambda_r \phi^2}{2}$.

In equilibrium finite temperature field theory we are interested in evaluating the finite temperature Green's functions,

$$G_T^{(n)}(x_1, x_2, \dots x_n) = \frac{Tr[\phi(x_1)\phi(x_2)\dots\phi(x_n)e^{-\beta H}]}{Tr[e^{-\beta H}]}. \quad (1.25)$$

Recall that $\beta = \frac{1}{kT}$, where T is the temperature. The temperature is a well defined quantity only for systems in equilibrium. In cosmology, we are frequently interested in systems which are in local thermal equilibrium. This happens when the interaction rates are high compared to the expansion rate of the universe. In such situations the above quantity is meaningful and of interest. There are of course, other situations such as when a phase transition occurs that the system is far from equilibrium and we will later discuss a formalism which allows us to keep track of the time evolution under these circumstance. For now, let us concentrate

on equilibrium finite temperature field theory. The fundamental quantity of interest for equilibrium finite temperature field theory is the partition function which is the generating function of Green's functions,

$$Z_\beta = \text{Tr}[e^{-\beta H}] \quad (1.26)$$

$$= \int D\phi \langle \phi | e^{-\beta H} | \phi \rangle. \quad (1.27)$$

Thus we need to evaluate $\langle \phi | e^{-\beta H} | \phi \rangle$. Recall that similar quantities,

$$\langle q'; t' | q; t \rangle = \langle q' | e^{-iH(t-t')} | q \rangle \quad (1.28)$$

are frequently evaluated in quantum mechanics. This points the way to using well developed techniques in zero temperature field theory for calculations in finite temperature field theory by making the identification $i\tau = -\beta$. Thus the inverse temperature is formally identified with the imaginary time direction and it is for this reason that the formalism described in this section is called the imaginary time formalism of finite temperature field theory.

An important point to notice is that since we are interested in evaluating the trace, the ' initial ' and ' final ' states have to be identified leading to periodicity in imaginary time τ for bosons. Fermions are anti-symmetric under the exchange of states and this leads to an anti-periodicity in imaginary time for the fermions.

Knowing the Hamiltonian allows one to extract the finite temperature Feynman rules which are then used to calculate the finite temperature effective potential using the tadpole method. Let us do this in some detail for the $\lambda\phi^4$ theory. The most important modification to the Feynman rules at finite temperature is in the propagator. In fact, the factors at the vertices and the combinatorics are the same

at finite temperature as they are at zero temperature. The important modification of the propagator comes essentially from the fact that we have definite periodicity in imaginary time for the bosons and anti-periodicity in imaginary time for the fermions. To determine the propagator it is sufficient to consider only the quadratic part of the action,

$$S_0[\phi] = \int_0^\beta d\tau d^3x \phi \left[-\frac{\partial^2}{\partial \tau^2} - \nabla^2 + M^2 \right] \phi. \quad (1.29)$$

Introducing the Fourier expansion for the field,

$$\phi(x, \tau) = \frac{1}{\beta} \sum_{m=-\infty}^{\infty} \int \phi_m(\underline{k}) e^{i(\underline{k} \cdot \underline{x} + \omega_{2m} \tau)} d^3k \quad (1.30)$$

one gets the propagator in momentum space to be,

$$D(\underline{k}, \omega_{2m}) = \frac{1}{\omega_{2m}^2 + k^2 + M^2} \quad (1.31)$$

where the quantity $\omega_{2m} = \frac{2\pi m}{\beta}$ results from the periodicity in imaginary time. Also notice that the periodicity in imaginary time leads to a discrete sum in the fourier transform. Thus,

$$\int \frac{d^4k}{(2\pi)^4} \rightarrow \frac{1}{\beta} \sum_m \int \frac{d^3k}{(2\pi)^3}. \quad (1.32)$$

For fermions we have anti-periodicity in imaginary time and thus instead of ω_{2m} , the quantity entering expressions is, $\omega_{2m+1} = \frac{\pi(2m+1)}{\beta}$.

Once again we will use the tadpole method, this time to compute the finite temperature corrections to the effective potential. The relation which gives the finite temperature correction, ΔV_T is,

$$\frac{d\Delta V_T}{d\omega} = \frac{1}{\beta} \sum_m \int \frac{d^3k}{(2\pi)^3} \frac{\lambda\omega}{\omega_{2m}^2 + k^2 + M_\phi^2(\omega)} \quad (1.33)$$

$$= \frac{1}{\beta} \sum_m \int \frac{d^3k}{(2\pi)^3} \ln \left[\frac{\omega_{2m}^2 + k^2 + M_\phi^2(\omega)}{\Lambda^2} \right]. \quad (1.34)$$

So far, the only thing that is really different from the zero temperature discussion has to do with the fact that there is a periodicity in imaginary time leading to a discrete fourier sum in the timelike direction. To proceed further, we choose to evaluate the sum first and then evaluate the integral. The sum can be evaluated by using the identity,

$$\sum_{m=-\infty}^{\infty} \ln(\omega_{2m}^2 + E_k^2) = \beta E_k + 2 \ln(1 - e^{-\beta E_k}) + (E_k \text{ independent terms}) \quad (1.35)$$

where

$$E_k^2 = k^2 + M_\phi^2, \quad \omega_{2m} = \frac{2\pi m}{\beta}. \quad (1.36)$$

Using the above identity we can evaluate the finite temperature correction to the effective potential,

$$\Delta V_T = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} E_k + \int \frac{d^3k}{(2\pi)^3} \ln [1 - e^{-\beta E_k}]. \quad (1.37)$$

At this point we note that the second term on the right is finite and the first term on the right hand side contains all the divergent part. Further, we note that in fact the divergence structure of this first term is identical to the divergence structure of the zero-temperature part of the effective potential. It thus follows that the counterterms that we introduced to absorb the divergences of the zero-temperature effective potential will be sufficient to absorb all the divergences of the finite temperature corrections to the effective potential. Hence, the net effect

of the finite temperature corrections is to add only the second term on the right hand side (which we will call ΔV_T^f to the expression for the renormalised effective potential. Let us evaluate the consequences of this term in some further detail. First, let us set $x = \beta k$ and do the solid angle integral. This yields,

$$\Delta V_T^f = \frac{1}{2\pi^2\beta^4} \int dx \, x^2 \ln \left[1 - e^{-(x^2 + \beta^2 M^2(\phi_c))^{\frac{1}{2}}} \right]. \quad (1.38)$$

For the region where $\frac{M}{T} \ll 1$ we can do a high temperature expansion to the above integral which yields,

$$\begin{aligned} \Delta V_T^f &= \frac{-\pi^2}{90} T^4 + \frac{M^2(\phi_c) T^2}{24} - \frac{T}{12\pi} (M^2(\phi_c))^{\frac{3}{2}} \\ &\quad - \frac{1}{64\pi} M^4(\phi_c) \ln \left[\frac{M^2(\phi_c)}{T^2} \right] + \frac{(3/2 - 2 \ln 4\pi - 2\gamma)}{64\pi^2} M^4(\phi_c) \\ &\quad + O(M^6\beta^2). \end{aligned} \quad (1.39)$$

To lowest order,

$$\Delta V_T^f = \frac{-\pi^2}{90} T^4 + \frac{M^2(\phi_c) T^2}{24}. \quad (1.40)$$

Thus, if we collect the coefficient of the ϕ_c^2 from the zero temperature potential and the finite temperature correction and define this coefficient as $m^2(T)$, then we get,

$$m^2(T) = m_{T=0}^2 + \frac{\lambda T^2}{24} \quad (1.41)$$

which indicates how a theory which has a broken symmetry at zero temperature ($m_{T=0}^2 < 0$) can have the symmetry restored at some higher temperature such that $m^2(T) = m_{T=0}^2 + \frac{\lambda T^2}{24} > 0$. Thus, the critical temperature will be given by $m^2(T_c) = 0 = m_{T=0}^2 + \frac{\lambda T_c^2}{24}$.

Thus we see how the imaginary time formalism of finite temperature field theory can be used to determine the critical temperature given a model of the interactions

of fields we are interested in, in terms of the parameters of the model. In this section we have used the simplest possible model of interactions to introduce the concepts, formalism and techniques of the imaginary time formalism of finite temperature field theory. In chapter 2 we will use this same formalism and these same techniques to study the finite temperature behaviour of a specific see-saw model of neutrino masses. As we will see the finite temperature behaviour of this model may have many interesting cosmological consequences including a possible explanation of the peak in comoving quasar space densities as well as providing a cosmological constant of the right magnitude to remove the discrepancy between the Hubble constant and the age of the universe. Of course, in addition to being of current interest this model will also allow us to illustrate the techniques and utility of the imaginary time formalism of finite temperature field theory. In particular, we will be able to analyze the manifold of degenerate vacua of the effective potential in that model to extract information about the nature of topological defects that will be formed at various temperatures.

1.3 The Real Time Formalism

So far we have studied the imaginary time formalism of finite temperature field theory. As already pointed out however, the imaginary time formalism cannot be used to study time dependent phenomena since the ‘ imaginary time ’ is identified with temperature. The best one can hope to do in this formalism is to stipulate that the temperature is some specified function of time $T(t)$. However, this implies that the system is always very close to some equilibrium configuration with a well defined temperature T . In particular, this implies that the density matrix is always

expressible in the form $\rho = e^{-\beta H}$. As we shall see during the transition period immediately following a phase transition the distribution function does not look like it can be fit by a purely thermal distribution function with a single well defined temperature at each instant of time.

The physical quantity best suited to describe the time evolution of a quantum statistical system out of equilibrium is the density matrix, $\rho(t)$. The density matrix can be expressed in terms of a complete set of eigenstates $|\psi_i\rangle$ in the following way, $\rho = \sum_i |\psi_i\rangle p_i \langle\psi_i|$ where p_i represents the probability of finding the state of the system in the state $|\psi_i\rangle$. Note that the density matrix need not necessarily be diagonal, however it must be hermitian and hence can be diagonalised by a unitary transformation and expressed in the form given above. The density matrix is particularly useful for describing the state of a system which is not a pure quantum mechanical state but is instead in a mixed state. In such cases the only information we may possess is the statistical information contained in the probabilities of finding the various different eigenstates of the system. An example of such a mixed statistical system is completely unpolarised light for which the density matrix would be an equally weighted sum over the different polarization states of light. Another example of a mixed statistical system which occurs frequently in nature and will be of concern to us is a system in thermal equilibrium at a given temperature T . For such a system all that we may know is that the probability of finding an eigenstate with energy E_j is proportional to $e^{-\frac{E_j}{kT}}$. For statistical systems we are frequently interested in the time evolution of a specific initial mixed state under the influence of a time dependent Hamiltonian.

The time evolution of the system is then determined by the quantum Liouville equation,

$$i\hbar \frac{\partial \rho}{\partial t} = [H(t), \rho(t)] \quad (1.42)$$

where $H(t)$ is the Hamiltonian of the system.

The formal solution of the quantum Liouville equation is given by,

$$\rho(t) = U(t, t_0) \rho(t_0) U^{-1}(t, t_0) \quad (1.43)$$

where $\rho(t_0)$ is the initial density matrix at time t_0 and $U(t, t_0)$ is the time evolution operator from t_0 to t .

Typically we are interested in systems that are initially in thermodynamic equilibrium with a well defined initial inverse temperature $\beta_i = \frac{1}{k_B T_i}$. The initial density matrix is then given by,

$$\rho(t_0) = e^{-\beta_i H_i}. \quad (1.44)$$

Notice that if we are interested in the ground state of the system which is the equilibrium state at zero temperature, we can isolate it by taking the limit $\beta_i \rightarrow \infty$.

The initial Hamiltonian H_i which governs the behavior of the system for times $t \leq t_0$ prepares the system to be in equilibrium with a well defined temperature and β_i upto t_0 . Interesting non-equilibrium phenomena will occur if for times $t > t_0$, the Hamiltonian becomes $H_{evol}(t)$ which is different from the initial Hamiltonian H_i . The time evolution of the expectation value of any operator \mathcal{O} is given by,

$$\langle \mathcal{O} \rangle(t) = \frac{Tr[e^{-\beta_i H_i} U^{-1}(t) \mathcal{O} U(t)]}{Tr[e^{-\beta_i H_i}]}. \quad (1.45)$$

The above expression can be put into a more compact form by noting once again that the time evolution operator, U also involves the exponential of the

Hamiltonian. In particular, for any time $T < t_0$,

$$U(T) = \exp[-iTH_i]. \quad (1.46)$$

We can use this fact to absorb the $\exp[-\beta_i H_i]$ factor into a modified time evolution operator. Let us re-express, $\exp[-\beta_i H_i] = \exp[-iH_i(T - i\beta_i - T)] = U(T - i\beta_i, T)$. Further, we insert the factor $U^{-1}(T)U(T) = 1$ in the numerator and using the fact that $U^{-1}(T)$ commutes with $\exp[-\beta_i H_i]$ we can re-express,

$$\langle \mathcal{O} \rangle(t) = \frac{\text{Tr}[U(T - i\beta_i, t)\mathcal{O}U(t, T)]}{\text{Tr}[U(T - i\beta_i, t)]}. \quad (1.47)$$

We have used slightly convoluted means to arrive at a compact result. In particular, we have hidden the initial density matrix but we have entered the complex time plane. Notice that in the imaginary time formalism we had only the imaginary time direction playing any role. In the real time formalism both the imaginary direction as well as the real direction enter the discussion. Thus the real time formalism should really be called the complex time formalism. However, that might have made the real time formalism sound even more dauntingly complex than it already appears. This may have impeded progress in this field for a few more years in real time.

We will use the real time formalism to study in detail the mechanism of dissipation via particle production[61, 62] in a later chapter. This is a process that is very important for understanding reheating following inflation. In chapter 3 we use the real time formalism to study in detail one of the most important time dependent and non-equilibrium phenomena associated with phase transitions. This is the phenomenon of dissipation of the initial vacuum energy of the field. The physical

mechanism for this is the particle production which takes place as the field oscillates about the minimum of its potential. We used the quantum Liouville equation to study the time evolution of the system starting with an initial configuration described by a density matrix, $\rho_{initial}$.

Once again the motivation for presenting this discussion in detail is twofold. First, it is a subject of current cosmological interest in its own right. Secondly it provides a detailed illustration of the techniques that will undoubtedly be useful in the study of a number of other time-dependent and non-equilibrium phenomena.

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Chapter 2

IMAGINARY TIME APPLICATION: A LATE TIME PHASE TRANSITION LINKED WITH MASSIVE NEUTRINOS

2.1 Introduction

One application of the imaginary time formalism that is of current cosmological interest is to the treatment of Late Time Phase Transitions(LTPT's). We will first describe what LTPT's are and then point out why we think they have a number of extremely interesting and potentially important cosmological consequences. Once this is done we will turn to a detailed discussion of a compelling particle physics model for such phase transitions. The detailed investigation of this model will also illustrate some of the strenghts of the imaginary time formalism for determining the existence of phase transitions, the critical temperature of the phase transition, the existence and nature of topological defects formed during the phase transition

in a given model for interacting fields.

Phase transitions that occur after the decoupling of matter and radiation have been discussed in the literature as Late Time Phase Transitions. The original motivation for considering LTPT's[11, 16, 17, 19, 20, 23] was the need to reconcile the extreme isotropy of the Cosmic Microwave Background Radiation (CMBR)[12] with the existence of large scale structure[13] and also the existence of quasars at high redshifts[14]. The hope that LTPT's would not disturb the CMBR was not realised[15]. In light of the fact that the anisotropy measurements of the CMBR are getting better[24], the distortions of the CMBR predicted by LTPT's may become important observational tests of these models.

In our mind the current motivation for a continued interest in LTPT's is twofold. First, LTPT's may provide an explanation for the peak observed in co-moving quasar space densities plotted as a function of redshift[1]. As we will see in detail in the discussion later, the most compelling particle physics model for LTPT's is the finite temperature behaviour of a see-saw model of neutrino masses. In such a model one finds a critical temperature which is a few times the relevant light neutrino mass. If we input into this model the neutrino masses required to solve the solar neutrino model by the MSW mechanism[3, 4] we obtain that the phase transition happens at the right epoch to explain the peak in quasar space densities. A second important motivation for a continued interest in the LTPT model being presented in this chapter is that it may help remove the discrepancy between the recently determination of the Hubble constant and the age of the universe.

LTPT's are used to generate density fluctuations either by bubble nucleation [16], dynamics of a slow rolling field[17] or by the formation of topological defects

such as domain walls(HSF). The appeal of LTPT's for structure formation lies in the fact that the density perturbations created are immediately non-linear and so will lead to much faster formation of structure. The models we will be considering in detail in this chapter involve the formation of topological defects as a result of LTPT's. The original idea of the production of soft domain walls as a result of an LTPT (HSF) has received several criticisms. A problem with the use of domain walls to form Large Scale Structure has been pointed out by Kolb and Wang[18]. They argue that domain walls may never form in a LTPT since thermal effects may not be able to drive different regions of the Universe to different parts of the disconnected vacuum manifold. However they also point out that domain walls would form if large fluctuations in the scalar field exist before the phase transition. One of the main criticisms of soft domain walls is that they may produce significant distortions of the CMBR. The most significant microwave distortion comes from collapsing domain wall bubbles[15]. These distortions produce hot and cold spots on $\sim 1^\circ$ angular scales and provide a signature for the formation of domain walls after recombination.

Since the collapse of closed domain wall bubbles to form black holes which then act as the central engine of quasars is one of the main themes of this part of this thesis, we now turn to a quick discussion of the distortions of the CMBR this produces. The temperature shift due to a photon traversing a collapsing domain wall bubble is

$$\frac{\Delta T}{T} = 2.64 \times 10^{-4} h^{-1} \beta A \sigma / (10 MeV^3) \quad (2.1)$$

where h , A , β are dimensionless numerical constants of order unity and σ is the surface tension of the domain wall. The present measurements of the CMBR

anisotropy then imply[37] that $\sigma < 0.5 \text{ MeV}^3$. This constraint, though important to keep in mind is not a problem for the viability of the model being presented in this chapter. We will return to this point after discussing our particle physics model for the LTPT.

2.2 Particle physics models for LTPT's

Discussions of realistic particle physics models capable of generating LTPT's have been carried out by several authors[20, 19]. It has been pointed out that the most natural class of models in which to realise the idea of LTPT's are models of neutrino masses with Pseudo Nambu Goldstone Bosons (PNGB's). The reason for this is that the mass scales associated with such models can be related to the neutrino masses, while any tuning that needs to be done can be protected from radiative corrections by the symmetry that gave rise to the Nambu-Goldstone modes[21].

Holman and Singh[23] studied the finite temperature behaviour of the see-saw model[22] of neutrino masses and found phase transitions in this model which result in the formation of topological defects. In fact, the critical temperature in this model is naturally linked to the neutrino masses.

To investigate in detail the finite temperature behaviour of models allowing for natural LTPT's we selected a very specific and extremely simplified version of the general see-saw model. However, we expect some of the qualitative features displayed by our specific simplified model to be at least as rich as those present in more complicated models.

We chose to study the 2-family neutrino model. Because of the mass hierarchy and small neutrino mixings[3] the expectation was to capture some of the essential

physics of the ν_e - ν_μ system in this way. The 2-family see-saw model we consider requires 2 right handed neutrinos N_R^i which transform as the fundamental of a global $SU_R(2)$ symmetry. This symmetry is implemented in the right handed Majorana mass term by the introduction of a Higgs field σ_{ij} , transforming as a symmetric rank 2 tensor under $SU_R(2)$ (both N_R^i and σ_{ij} are singlets under the standard model gauge group). The spontaneous breaking of $SU_R(2)$ via the vacuum expectation value (VEV) of σ gives rise to the large right handed Majorana masses required for the see-saw mechanism to work. Also, the spontaneous breaking of $SU_R(2)$ to $U(1)$ gives rise to 2 Nambu Goldstone Bosons. The $SU_R(2)$ symmetry is explicitly broken in the Dirac sector of the neutrino mass matrix, since the standard lepton doublets l_L and the Higgs doublet Φ are singlets under $SU_R(2)$. It is this explicit breaking that gives rise to the potential for the Nambu Goldstone modes via radiative corrections due to fermion loops. Thus, these modes become Pseudo Nambu Goldstone Bosons (PNGB's).

The relevant Yukawa couplings in the leptonic sector are:

$$-\mathcal{L}_{\text{yuk}} = y_{ai} \bar{l}_L^a N_R^i \Phi + y \bar{N}_R^i N_R^j \sigma_{ij} + \text{h.c.} \quad (2.2)$$

where $a, i, j = 1, 2$. The $SU_R(2)$ symmetry is implemented as follows:

$$\begin{aligned} N_R^i &\rightarrow U_j^i N_R^j \\ \sigma_{ij} &\rightarrow U_i^k \sigma_{kl} (U^T)_j^l \end{aligned} \quad (2.3)$$

where U_j^i is an $SU_R(2)$ matrix. The first (Dirac) term breaks the symmetry explicitly.

We can choose the VEV of σ to take the form[25]: $\langle \sigma_{ij} \rangle = f \delta_{ij}$, thus breaking $SU_R(2)$ spontaneously down to the $U(1)$ generated by τ_2 (where τ_i are the

Pauli matrices). This symmetry breaking gives rise to the 2 PNGB's, whose finite temperature effective potential is of interest.

After the Higgs doublet acquires its VEV, we have the following mass terms for the neutrino fields:

$$-\mathcal{L}_{\text{mass}} = m_{ai} \bar{\nu}_L^a N_R^i + M \bar{N}_R U U^T N_R^c + \text{h.c.} \quad (2.4)$$

where ν_L^a are the standard neutrinos, $m_{ai} = y_{ai} v/\sqrt{2}$, $M = yf/\sqrt{2}$.

Examining the expression for $-\mathcal{L}_{\text{mass}}$ above we notice that it may be expressed more compactly by introducing the fields,

$$N'_R = U^\dagger N_R \quad (2.5)$$

Then the expression for $-\mathcal{L}_{\text{mass}}$ becomes,

$$-\mathcal{L}_{\text{mass}} = m_{ai} \bar{\nu}_L^a U_j^i N'^j_R + M \bar{N}'_R N'^c_R + \text{h.c.} \quad (2.6)$$

At this point we can introduce the ν - mass matrix to rewrite the above terms in a block matrix form :

$$-\mathcal{L}_{\text{mass}} = (\bar{\nu}_L \quad \bar{N}_R) \begin{pmatrix} 0 & mU \\ (mU)^T & M \end{pmatrix} \begin{pmatrix} \nu_L^c \\ N_R^c \end{pmatrix} + \text{h.c.} \quad (2.7)$$

where the mass matrix, \mathcal{M}_ν ,

$$\mathcal{M}_\nu = \begin{pmatrix} 0 & mU \\ (mU)^T & M \end{pmatrix} \quad (2.8)$$

can be re-expressed in the suggestive form,

$$\mathcal{M}_\nu = \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix} + \begin{pmatrix} 0 & mU \\ (mU)^T & 0 \end{pmatrix} \quad (2.9)$$

In particular, in the physical case where $|m_{ai}| \ll M$ we can use perturbation theory techniques to diagonalise the mass matrix. Specifically, we need to find the eigenvalues of the mass matrix by using the second matrix as a perturbation on the first matrix in the expression above. Thus, the zeroth order eigenvalues are $0, 0, M, M$ and the corresponding eigenvectors can be taken to be of the block form $\begin{pmatrix} e_1 \\ 0 \end{pmatrix}, \begin{pmatrix} e_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_1 \end{pmatrix}, \begin{pmatrix} 0 \\ e_2 \end{pmatrix}$ where

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.10)$$

The most significant effects due to higher order corrections will clearly be to the first two eigenvalues which are both 0 at the zeroth order in perturbation theory. Let us calculate the corrections to these two eigenvalues explicitly. Call the perturbation matrix M_1 i. e.

$$M_1 = \begin{pmatrix} 0 & mU \\ (mU)^T & 0 \end{pmatrix} \quad (2.11)$$

Then, the first order corrections to the eigenvalues λ_i for $i = 1, 2$ is given by,

$$\lambda_i^{(1)} = \langle i | M_1 | i \rangle \quad (2.12)$$

$$= \begin{pmatrix} e_i & 0 \end{pmatrix} \begin{pmatrix} 0 & mU \\ (mU)^T & 0 \end{pmatrix} \begin{pmatrix} e_i \\ 0 \end{pmatrix} \quad (2.13)$$

$$= 0 \quad (2.14)$$

Thus the first non-zero correction to these eigenvalues occur at second order in perturbation theory which is given by,

$$\lambda_i^{(2)} = \sum_{j \neq i} \frac{\langle i | M_1 | j \rangle \langle j | M_1 | i \rangle}{\lambda_i^{(0)} - \lambda_j^{(0)}} \quad (2.15)$$

Substituting the explicit form for M_1 and the eigenvectors in the above expression we get,

$$\lambda_1^{(2)} = -\frac{1}{M} \left[(mU)_{11}^2 + (mU)_{12}^2 \right] \quad (2.16)$$

and

$$\lambda_2^{(2)} = -\frac{1}{M} \left[(mU)_{21}^2 + (mU)_{22}^2 \right] \quad (2.17)$$

where

$$(mU)_{aj} = m_{ai} U_j^i \quad (2.18)$$

$$= m_{ai} \left(\exp(i \underline{\xi} \cdot \underline{\tau} / f) \right)_j^i \quad (2.19)$$

For the case we are interested here U belongs to $SU_R(2)$ and can be expressed in the convenient form,

$$U = \exp[i \underline{\xi} \cdot \underline{\tau} / f] \quad (2.20)$$

$$= \cos ||\xi|| + i \hat{\xi} \cdot \underline{\tau} \sin ||\xi|| \quad (2.21)$$

where $||\xi|| = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} / f$ and $\hat{\xi}_i = \xi_i / \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$.

Recall that we chose the symmetry breaking scheme where $SU_R(2)$ was spontaneously broken down to the $U(1)$ generated by τ_2 . Thus the Nambu-Goldstone modes point in the τ_1 and τ_3 directions. The most general form of U that we will therefore be interested in therefore is of the form,

$$U = \exp[i(\xi_1 \tau_1 + \xi_3 \tau_3) / f] \quad (2.22)$$

In particular,

$$U_1^1 = \cos ||\xi|| + i\widehat{\xi}_3 \sin ||\xi|| \quad (2.23)$$

$$U_2^1 = i\widehat{\xi}_1 \sin ||\xi|| \quad (2.24)$$

$$= U_1^2 \quad (2.25)$$

$$U_2^2 = \cos ||\xi|| - i\widehat{\xi}_3 \sin ||\xi|| \quad (2.26)$$

If for concreteness and simplicity we take $m_{ai} = m \delta_{ai}$ then,

$$\lambda_1^{(2)} = -\frac{m^2}{M} \left[\cos(2||\xi||) + i\widehat{\xi}_3 \sin(2||\xi||) \right] \quad (2.27)$$

and

$$\lambda_2^{(2)} = -\frac{m^2}{M} \left[\cos(2||\xi||) - i\widehat{\xi}_3 \sin(2||\xi||) \right] \quad (2.28)$$

Since the effective potential for the ξ_i 's is generated by the propagation of neutrinos in the loops the effect of the heavier neutrinos is suppressed by extra powers of m/M . Thus, it is the two lightest neutrinos that contribute most significantly to the effective potential for the ξ 's.

Substituting the above eigenvalues back in the expression for $-\mathcal{L}_{\text{yuk}}$ and recalling that,

$$\psi_L = \frac{1}{2}(1 + \gamma_5)\psi \quad (2.29)$$

$$\psi_R = \frac{1}{2}(1 - \gamma_5)\psi \quad (2.30)$$

we get that,

$$\begin{aligned}
-\mathcal{L}_{\text{yuk}} = & -\frac{m^2}{M}\bar{\psi}_1 \left[\cos(2||\xi||) + i\hat{\xi}_3 \sin(2||\xi||)\gamma_5 \right] \psi_1 \\
& -\frac{m^2}{M}\bar{\psi}_2 \left[\cos(2||\xi||) - i\hat{\xi}_3 \sin(2||\xi||)\gamma_5 \right] \psi_2
\end{aligned} \tag{2.31}$$

2.3 The effective potential for our model and its implications

We will treat the ξ_i 's as classical background fields and not allow them to propagate in loops. Thus, the effective potential of the ξ_i 's is going to be due to the fermion loops. Thus when we perform the shift $\xi \rightarrow \xi - w$ we need only keep the lowest order term in ξ . (The terms that will contribute to the tadpole diagram have to be of the form $\bar{\psi}\xi\psi$ and can be isolated from the above expression for \mathcal{L}_{yuk} by performing the shift $\xi \rightarrow \xi - w$.)

When we examine the expression for \mathcal{L}_{yuk} above we notice that on performing the shift $\xi \rightarrow \xi - w$ we will need to expand the terms $||\xi - w||$ and $\frac{\xi_3 - w_3}{||\xi_3 - w_3||}$. To linear order in ξ , their expansions are,

$$||\xi - w|| = ||w|| - \frac{\underline{\xi} \cdot \underline{w}}{||w||} + O(\xi^2) \tag{2.32}$$

$$\frac{\xi_3 - w_3}{||\xi_3 - w_3||} = -\widehat{w}_3 + \frac{\xi_3}{||w||} - w_3 \frac{\underline{\xi} \cdot \underline{w}}{||w||^3} + O(\xi^2) \tag{2.33}$$

Using these expansions in \mathcal{L}_{yuk} and performing straightforward trigonometric and algebraic manipulations one can extract the following $\bar{\psi}\xi\psi$ couplings that are all relevant for computing the tadpole diagrams that will give us an expression for the effective potential. The coefficient of the $\bar{\psi}_1 \xi_3 \psi_1$ term is: $-\frac{m^2}{M} [y_1 + i\gamma_5 y_2]$, where,

$$y_1 = \frac{2w_3 \sin(2||w||)}{||w||} \quad (2.34)$$

$$y_2 = \frac{w_3^2 \cos(2||w||)}{||w||^2} + \left(\frac{1}{||w||} - \frac{w_3^2}{||w||^3} \right) \sin(2||w||) \quad (2.35)$$

The coefficient of the $\overline{\psi}_1 \xi_1 \psi_1$ term is: $-\frac{m^2}{M} [y_3 + i\gamma_5 y_4]$, where,

$$y_3 = \frac{2w_1 \sin(2||w||)}{||w||} \quad (2.36)$$

$$y_4 = \frac{w_3 w_1 \cos(2||w||)}{||w||^2} - \frac{w_3 w_1}{||w||^3} \sin(2||w||) \quad (2.37)$$

The coefficient of the $\overline{\psi}_2 \xi_1 \psi_2$ term is $-\frac{m^2}{M} [y_3 - i\gamma_5 y_4]$ and the coefficient of the $\overline{\psi}_2 \xi_3 \psi_2$ term is $-\frac{m^2}{M} [y_1 - i\gamma_5 y_2]$.

The tadpole diagram with one external ξ_1 leg will receive contributions from the loops due to ψ_1 and ψ_3 and is given by,

$$\frac{dV_{eff}}{dw_1} = \Gamma_{w_1, w_3}^{(1,0)} \quad (2.38)$$

$$= -8i \int \frac{d^4 k}{(2\pi)^4} \frac{m^2}{M} \frac{[y_3 A_1 - i y_4 B_1]}{k^2 - (A_1^2 - B_1^2)} \quad (2.39)$$

where y_3, y_4 were given earlier and A_1, B_1 are given by:

$$A_1 = -\frac{m^2}{M} \cos(2||w||) \quad (2.40)$$

$$B_1 = \frac{im^2}{M} \left[\frac{w_3}{||w||} \sin(2||w||) \right] \quad (2.41)$$

Similarly the tadpole diagram with one external ξ_3 leg will be given by,

$$\frac{dV_{eff}}{dw_3} = \Gamma_{w_1, w_3}^{(0,1)} \quad (2.42)$$

$$= -8i \int \frac{d^4k}{(2\pi)^4} \frac{m^2}{M} \frac{[y_1 A_1 - i y_2 B_1]}{k^2 - (A_1^2 - B_1^2)} \quad (2.43)$$

The above expressions can be integrated to yield,

$$V_{eff} = 4i \int \frac{d^4k}{(2\pi)^4} \ln \left(\frac{k^2 - \mathcal{M}^2(\xi)}{\Lambda^2} + i\epsilon \right) \quad (2.44)$$

where,

$$\mathcal{M}^2 = \frac{m^4}{M^2} (\cos^2 2||\xi|| + \hat{\xi}_3^2 \sin^2 2||\xi||) \quad (2.45)$$

The above expression can be explicitly evaluated by performing a rotation to Euclidean space to yield,

$$V_{eff}(\xi_1, \xi_3) = V_{eff}(\mathcal{M}^2) \quad (2.46)$$

$$= \frac{4}{64\pi^2} \left[\frac{\Lambda^4}{2} + 2\Lambda^2 \mathcal{M}^2 + \mathcal{M}^4 \left(\ln \frac{\mathcal{M}^2}{\Lambda^2} - \frac{1}{2} \right) \right] \quad (2.47)$$

where as always Λ is the cut-off introduced to regularise the integral.

In order to look for phase transitions in this system we should now add in the effects of the heat bath. This should be done with some caution. The reason for this is that by the time of decoupling, the neutrinos have long since dropped out of thermal equilibrium with the ambient heat bath, which would seem to indicate that the finite temperature formalism is inapplicable. However, we should recall that despite being out of thermal equilibrium, neutrinos preserve a thermal-like distribution after they decouple (just as the microwave background photons do),

with a temperature inversely proportional to the scale factor[26]. Thus, we can still compute the finite temperature effective potential and use it as a diagnostic of critical behavior, as long as we recall that the relevant temperature is that of the neutrinos and not that of the heat bath. Another caveat is that once the temperature is comparable to the mass of the neutrino, the distribution function will cease to look thermal.

The finite temperature effects can be studied by calculating the finite temperature corrections to the effective potential which is given by,

$$\Delta V_T = \frac{4T^4}{2\pi^2} \int_0^\infty dx \, x^2 \ln \left[1 + \exp[-\sqrt{x^2 + \frac{\mathcal{M}^2}{T^2}}] \right] \quad (2.48)$$

Recall that we parametrized the PNGB's as ξ_1 and ξ_3 and defined the quantity \mathcal{M} to be:

$$\mathcal{M}^2 = \frac{m^4}{M^2} (\cos^2 2||\xi|| + \hat{\xi}_3^2 \sin^2 2||\xi||) \quad (2.49)$$

Performing the high temperature expansion of the complete potential and discarding terms of order $(\mathcal{M}^2)^3/T^2$ or higher we get,

$$V_{\text{tot}}(\xi_1, \xi_3) = V(\mathcal{M}^2) = (V_0 - \frac{7\pi^2 T^4}{90}) + (m_r^2 + T^2/6)\mathcal{M}^2 + \frac{(\mathcal{M}^2)^2}{8\pi^2} (n - \log \frac{T^2}{\mu^2}), \quad (2.50)$$

where $n = 2\gamma - 1 - 2 \log \pi \sim -2.1303$, m_r is a parameter in the model and μ is the renormalisation scale. \mathcal{M} is naturally of the neutrino mass scale in this model.

A study of the manifold of degenerate vacua of the effective potential at different temperatures revealed phase transitions in this model accompanied by the formation of topological defects at a temperature of a few times the relevant neutrino mass . Typically at higher temperatures the manifold of degenerate vacua consisted of a set of disconnected points whereas at lower temperatures the man-

ifold was a set of connected circles. Thus, domain walls would form at higher temperatures which would evolve into cosmic strings at lower temperatures.

An important point is that the potential depends on the ξ 's only through \mathcal{M}^2 . This has the effect of taking a two dimensional problem and turning it into a one dimensional one, i. e. we need only find the value of \mathcal{M}^2 that minimizes the effective potential. A picture of the potential for $m_r^2 = 1$ is given in fig. 1.

We can understand some features of the manifold of degenerate vacua of this effective theory if we first uncover some of the symmetries of the potential. Note that \mathcal{M}^2 is left invariant by the transformations:

$$\begin{aligned} ||\xi|| &\rightarrow ||\xi'|| = ||\xi|| + \frac{n\pi}{2} \\ \frac{\xi_i}{||\xi||} &\rightarrow \frac{\xi_i'}{||\xi'||} = \frac{\xi_i}{||\xi||} \end{aligned} \quad (2.51)$$

where n is an arbitrary integer. Furthermore if we define $\widehat{\mathcal{M}}^2 \equiv \cos^2 2||\xi|| + \widehat{\xi}_3^2 \sin^2 2||\xi||$, we see that $\widehat{\mathcal{M}}^2$ is restricted to lie between 0 and 1. It vanishes when $\xi_3 = 0, \xi_1 = (2k+1)\pi/4$, while it attains its maximum value of 1 when *either* $\xi_1 = 0$ or $||\xi|| = k\pi/2$ ($k \in \mathcal{Z}$). Thus, if $\widehat{\mathcal{M}}^2 = 0$ is the ground state, then the vacuum manifold consists of a discrete set of points, while if $\widehat{\mathcal{M}}^2 = 1$ minimizes the potential, then this manifold is the set theoretic union of an infinite number of circles with discrete radii and the line $\xi_1 = 0$, which intersects all of these circles. We should note that due to the above symmetries some of these vacua will become identified. For example, in the $\widehat{\mathcal{M}}^2 = 0$ case, the only *distinct* vacua are: $\xi_3 = 0$ and $\xi_1 = (2k+1)\pi/4$, with $k = 0, 1, 2, 3$. For the $\widehat{\mathcal{M}}^2 = 1$ situation the distinct vacua are $\xi_1 = 0$ or $||\xi|| = k\pi/2$, $k = 0, 1, 2, 3$.

At this point we can define $m_\nu = m^2/M$ and scale all dimensionful quantities with this mass. Thus we define $\widehat{T} = T/m_\nu$, $\widehat{m}_r = m_r/m_\nu$ and we set the renor-

malization scale μ equal to m_ν (which, of course, does not affect any physics). We then consider the dimensionless effective potential

$$\begin{aligned}\mathcal{V}(\xi_1, \xi_3) &\equiv m_\nu^{-4}(V_{\text{tot}}(\xi_1, \xi_3) - V_{\text{tot}}(\xi_1 = \frac{\pi}{4}, \xi_3 = 0)) \\ &= A(\hat{T})\widehat{\mathcal{M}}^2 + \frac{B(\hat{T})}{8\pi^2}(\widehat{\mathcal{M}}^2)^2\end{aligned}\tag{2.52}$$

where $A(\hat{T}) = \widehat{m}_r^2 + \hat{T}^2/6$, $B(\hat{T}) = n - \log \hat{T}^2$.

We can now examine the phase structure of \mathcal{V} as a function of temperature. The first fact we can prove is that if $m_r^2 \geq 0$, then the ground state is always given by $\widehat{\mathcal{M}}^2 = 0$. This is easiest to see by completing the square in \mathcal{V} to write it as:

$$\mathcal{V}(\xi_1, \xi_3) = -2\pi^2 \frac{A(\hat{T})^2}{B(\hat{T})} + \frac{B(\hat{T})}{8\pi^2}(\widehat{\mathcal{M}}^2 + 4\pi^2 \frac{A(\hat{T})}{B(\hat{T})})^2\tag{2.53}$$

If \hat{T}_0 is the zero of $B(\hat{T})$ ($\hat{T}_0 = \exp(n/2) \sim 0.34$), then for $\hat{T} > \hat{T}_0$, the quantity $g(\hat{T}) = -4\pi^2 A(\hat{T})/B(\hat{T})$ is strictly greater than 2, while for $\hat{T} < \hat{T}_0$ $g(\hat{T})$ is negative. Using these facts, as well as the restriction that $\widehat{\mathcal{M}}^2$ lies between 0 and 1 in plotting the parabola described by \mathcal{V} as a function of $\widehat{\mathcal{M}}^2$ proves our claim. Thus domain walls appear when the ξ_i 's roll to their minima in the $m_r^2 \geq 0$ case (if initial condidtions allow for this; see below).

The $m_r^2 < 0$ case is more interesting. Set $\alpha = -6m_r^2/\hat{T}_0^2$. In order that the high temperature approximation remain valid during our analysis, we restrict ourselves to values of \hat{T} satisfying $\hat{T} \gg 1$.

First consider the $\alpha \leq 1$ case. Then for the values of \hat{T} under consideration, $A(\hat{T}) > 0$, $B(\hat{T}) < 0$. Furthermore, the location of the peak of \mathcal{V} is at $\widehat{\mathcal{M}}_{\text{max}}^2 \sim 0.77(x - \alpha)/\log x$, where $x = (\hat{T}/\hat{T}_0)^2$. Now, $\hat{T} \gg 1$ implies that we must only consider values of x larger than $\hat{T}_0^{-2} \sim 8.5$. In this regime, $\widehat{\mathcal{M}}_{\text{max}}^2$ is always larger than one, so that again, a plot of the parabola described by \mathcal{V} as a function of $\widehat{\mathcal{M}}^2$

leads to the conclusion that the ground state is at $\widehat{\mathcal{M}}^2 = 0$, just as in the $m_r^2 \geq 0$ case.

What happens in the $\alpha > 1$ case? For $\alpha \ll \widehat{T}_0^{-2}$, we are essentially back to the situation described in the previous paragraph. However, suppose that α is comparable to or larger than \widehat{T}_0^{-2} . Then it is simple to see (using once more the parabola described by \mathcal{V} as a function of $\widehat{\mathcal{M}}^2$, see fig. (2)) that the ground state switches from $\widehat{\mathcal{M}}^2 = 0$ to $\widehat{\mathcal{M}}^2 = 1$ at a critical temperature \widehat{T}_c such that $-8\pi^2 A(\widehat{T}_c)/B(\widehat{T}_c) = 1$. This is a second order phase transition which takes the domain walls present when $\widehat{\mathcal{M}}^2 = 0$ was the ground state and converts them into cosmic strings which are allowed when $\widehat{\mathcal{M}}^2 = 1$ is the vacuum state. This provides a natural solution to the problem of having a long-lived network of domain walls in theories of late time phase transitions. We display some snapshots of the phase transition in fig. (3).

We see then that, as stated in the introduction, the vacuum structure of this class of models is surprisingly rich in terms of the evolution of topological defects. We can also find large regions of parameter space for which the critical temperature of both the phase transition that generates the domain walls as well as of the one that converts them into strings is $\mathcal{O}(10m_\nu)$. Thus these models are well suited to being used for late time phase transition purposes.

Let us now return to considering the observational data that provides the strongest motivation for a continued investigation of LTPT's. It has recently been pointed out by M. Schmidt et al. that comoving quasar space densities exhibit a strong peak at redshifts of 2 to 3[1]. In their chapter they plot quasar space density as a function of redshift (z) and also separately as a function of cosmic time. There is an unmistakable peak in the quasar space density in the region of

redshift 2 to 3. The authors further point out that the observed decline in space density for space density for $z > 3$ is not a result of instrumental difficulties in detecting distant quasars. On the contrary, the decline in observed quasar density is a real decline in the number density for $z > 3$ [2].

Thus, Schmidt et al. point out that quasar density peaks sharply at an epoch of about 2.3 billion years after the Big Bang. The full width at half maximum of the peak is around 1.4 billion years. Their discussion makes the final point that one needs to understand these time scales in terms of the formation and evolution of quasars.

What we would like to argue here is that such a peaked distribution of objects may result from the formation of topological defects as the universe goes through a phase transition. Such topological defects could form the seeds around which quasars light up.

The central power supply of quasars is believed to be gravitational in origin[8]. It is suggested in this chapter that this central power supply may be formed as a result of topological defect formation in a phase transition linked with massive neutrinos. Such a phase transition would happen at the right epoch if one believes the neutrino masses implied by the MSW solution to the solar neutrino problem. The production of black holes as a result of cosmological phase transitions has been discussed by various authors[5, 6, 7].

The idea of topological defects as seeds for structure formation has been around for some time. There are many excellent and recent reviews on the subject[9, 10]. The idea that topological defects formed after the decoupling of matter and radiation may play an important role in structure formation has also been discussed before by Hill, Schramm and Fry[11](HSF).

In the Standard Cosmology the redshift range of 2 to 3 corresponds to a few meV in mass scales[26]. Thus the critical temperature of the phase transition required to produce quasars is fixed at a few meV.

As we have already pointed out in the context of our particle physics model for LTPT's we expect a phase transition with a critical temperature of a few times the relevant light neutrino mass. Let us therefore now turn to a discussion of what evidence we have for non-zero masses for neutrinos and what masses for light neutrinos are implied by these observations.

At neutrino detectors around the world, fewer electron neutrinos are received from the sun than predicted by the Standard Solar Model. An explanation of the deficiency is offered by the MSW mechanism[3] which allows the ν_e produced in solar nuclear reactions to change into ν_μ . This phenomenon of neutrino mixing requires massive neutrinos with the masses for the different generations different from each other[3].

The model we considered earlier was an extremely simple one. Although it had 2 families of light neutrinos, there was only one single light neutrino mass. As such this model was not compatible with the MSW effect. However it is fairly straightforward to modify our original model to make it compatible with the MSW effect as is shown in what follows.

To ensure that it is not possible to choose the weak interaction eigenstates to coincide with the mass eigenstates we must require the 2 neutrino mass scales to be different. We can ensure neutrino mixing in our model by demanding that m_{ai} be such that $m_{11} \neq m_{22}$ and $m_{12} = 0 = m_{21}$. In this case, the effective potential $V_{\text{tot}}(\xi_1, \xi_3) = 1/2(V(\mathcal{M}_1^2) + V(\mathcal{M}_2^2))$ with $V(\mathcal{M}_i^2)$ having the same functional form

as $V(\mathcal{M}^2)(i = 1, 2)$ and \mathcal{M}_i^2 given by the following expression:

$$\mathcal{M}_i^2 = \frac{m_{ii}^4}{M^2}(\cos^2 2||\xi|| + \hat{\xi}_3^2 \sin^2 2||\xi||) \quad (2.54)$$

.

Further, if $m_{11} \ll m_{22}$ then $V_{\text{tot}}(\xi_1, \xi_3) = V(\mathcal{M}_2^2)/2$, which is exactly half the finite temperature effective potential we discussed earlier except the neutrino mass scale is the heavier neutrino mass scale. Hence, the discussion on phase transitions and formation of topological defects we carried out earlier goes through exactly except that the critical temperature is determined by the mass scale of the heavier of the 2 neutrinos.

In the complete picture of neutrino masses[3], the neutrinos might have a mass hierarchy analogous to those of other fermions. Further, we expect that the mixing between the first and third generation might be particularly small . Thus we will only consider ν_e - ν_μ mixing. The data seems to imply a central value for the mass of the muon neutrino to be a few meV[27].

At this point, let us re-examine the constraint on the domain wall tension, σ , placed by the measurements of the CMBR on small angular scales. Recall that the constraint is $\sigma < 0.5 MeV^3$. An estimate of σ in terms of the quantities m_ν and f introduced in our model can be obtained[7]. (To make contact with the work of L. Widrow cited above please note that his $\lambda m^4 = m^4(\nu_\mu)$ and $m = f$ in our notation.) Thus, the constraint on σ then implies that $f < 10^{15} GeV$. Our model is clearly an effective theory with f being some higher symmetry breaking scale on which it is tough to get an experimental handle. However, the constraint derived above is in fact natural in the context of the see-saw model of neutrino masses embedded in Grand Unified Theories as discussed by Mohapatra and Parida[28]

and also by Deshpande, Keith and Pal[29].

There are a few points on which we'd like to add some further comments. The first has to do with black hole formation as a result of LTPT's. The second with the possible role of a LTPT linked with the mass of ν_τ in the formation of LSS (on scales $\sim 100Mpc$). Finally, a brief comment on the post COBE status of LTPTs is made.

A number of different groups have studied different mechanisms for black hole formation as a result of cosmological phase transitions. Black hole formation as a result of bubble wall collisions has been suggested by Hawking et al[30]. Collapse of trapped false vacuum domains to produce black holes has been studied by Kodama et al.[5] and Hsu[31]. The collapse of closed domain walls to form black holes has been studied by Ipser and Sikivie[32] and also by Widrow[7]. In fact, Widrow has suggested that the collapse of closed domain walls is likely to produce black holes as a result of LTPTs. An estimate of the mass of black holes produced in LTPTs is $M_{BH} \sim 10^9 M_{solar} (\frac{R}{10^3 Mpc})^2 (\frac{f}{10^8 GeV})$, where R is the radius of the closed domain wall and we have taken $m(\nu_\mu)$ to be $3meV$ to obtain our estimate.

There are of course two definite uncertainties in getting a number out of the above expression. First, there will obviously be a range of possible sizes of closed domain wall bubbles that will be formed in the transient period of the phase transition. A natural upper limit to the size of the bubble will be the horizon size at the epoch of the phase transition. Thus this implies that $R < 10^3 Mpc$. The horizon size at the onset of the phase transition is smaller. Further, it should be noted that sub-horizon sized bubbles which will be formed in the transient period of the phase transition may also play an important role. Secondly, as already pointed out our model is clearly an effective theory with f being some higher symmetry

breaking scale on which it is tough to get an experimental handle.

A more detailed analysis of the masses of BH produced in LTPTs as well as the number densities of these BH will be the subject of a later work. For now, the fact that it is possible to produce BH which may act as the central engine of quasars is pointed out. One can check that the mechanism of collapse of domain wall bubbles can in fact give roughly the needed number of black holes to power quasars. The Hubble radius at the onset of the phase transition at $z \sim 5$, is a few hundred Mpc . One expects a distribution of bubble sizes with the horizon as an upper limit to be formed during a cosmological phase transition. In fact, Turner, Weinberg and Widrow[33] have carried out a detailed study of the distribution of bubble sizes resulting from cosmological phase transitions. They report that typical bubble sizes in a successful phase transition range from 0.01 to 1 times the Hubble radius at the epoch of the phase transition and depends only very weakly on the energy scale of the phase transition. Thus, the most massive black holes formed from the phase transition being studied here will have an abundance $\sim 10^{-5}$ to $10^{-6} Mpc^{-3}$ with a higher abundance of less massive black holes. These numbers, in fact match very well with the observational number densities of quasars as discussed by Warren and Hewett[34], and by Boyle et al. and Irwin et al[35]. The point is that the epoch of the phase transition linked with massive neutrinos is about right to explain the observed peak in the quasar distribution. Further, order of magnitude estimates of the masses of black holes produced and their abundances are consistent with observations within the uncertainties discussed above. A detailed analysis of the efficiency of black hole formation and therefore an accurate number for the space density of black holes produced in this model will be carried out in a later work.

The other point that needs to be commented on is the role of a massive ν_τ in

the scheme of things presented here. The mass of ν_τ is less well determined but one would expect an earlier phase transition with a T_c of a few times $m(\nu_\tau)$. In fact, such a phase transition may well be responsible for the LSS seen on the scale of $\sim 100 Mpc$. There already exists a great deal of discussion on this subject.

However, black hole formation in this earlier phase transition would be more difficult to observe. In fact, it is the black holes that are formed in the most recent phase transition that will have the greatest observable consequences.

The anisotropy of the CMBR on large angular scales is more closely linked to the formation of LSS (on scales ~ 100 Mpc). One can relate the power spectrum of density fluctuations to the gravitational potential power spectrum responsible for distortions of the CMBR. This link is thoroughly discussed by Jaffe, Stebbins and Frieman (JSF)[36]. Since considerable processing of the power spectrum must take place in the process of black hole formation a relationship between quasar distribution and the distortions of the CMBR on large angular scales is considerably more difficult to establish. The viability of LTPTs in the post-COBE era has recently been discussed by Schramm and Luo[37]. They point out that LTPTs are still a viable model for the formation of LSS.

To place things in perspective one should keep in mind that different mechanisms may play important roles in structure formation at different length scales. This point has been emphasised by Carr[38] who has given a comprehensive discussion on the origin of cosmological density fluctuations. In fact, JSF also conclude their discussion by pointing out that the final power spectrum is most likely due to a combination of primordial and late time effects.

2.4 Conclusion

This chapter provides a detailed application of the imaginary time formalism of finite temperature field theory. In addition to providing an illustration of many useful techniques this is also of current cosmological interest because of a number of recent observations and measurements that it has the promise of tying up. Thus, the MSW solution to the solar neutrino problem seems to imply a muon neutrino mass of a few meV. This in turn would lead to a phase transition in the PNGB fields associated with massive neutrinos with a critical temperature of several meV. This phase transition happens at the correct epoch in the evolution of the universe to provide a possible explanation of the peak in quasar space density at redshifts of 2 to 3. This work is clearly only a first step in bringing these ideas together. There are clearly some issues which need to be addressed in greater depth and explored in fuller detail. Thus, a more detailed and accurate analysis of the black hole formation efficiency and the masses and number densities of BH's needs to be carried out.

To obtain more precise estimates of the black hole number densities and masses one needs to make a more detailed analysis of domain formation and growth in FRW cosmologies. Domain formation and growth are among the most important non-equilibrium phenomena associated with phase transitions. The real time formulation of finite-temperature field theory is the most logical and well-suited formalism to study such phenomena.

Boyanovsky, Lee and Singh have studied domain formation and growth and discussed it at length using the real time time formalism[16]. This study was carried out in Minkowski space. We need to extend this work to FRW cosmologies.

In fact, non-equilibrium phenomena such as particle creation, entropy growth and dissipation have already been studied for FRW cosmologies by Boyanovsky, de Vega and Holman[40]. We now need to examine domain formation and growth in this formalism.

This number density and mass distribution of BH's then needs to be compared to the observational data on quasars. Further, a more detailed analysis of the angular dependence of the CMBR anisotropy produced needs to be carried out and compared to observations.

Another interesting cosmological implication of the calculation outlined above is the potential it has of resolving the discrepancy between the recently measured values of the Hubble constant and the age of the universe. Increasing improvements in the independent determinations of the Hubble constant and the age of the universe now seem to indicate that we need a small non-vanishing cosmological constant to make the two independent observations consistent with each other. The cosmological constant can be physically interpreted as due to the vacuum energy of quantized fields. To make the cosmological observations consistent with each other we would need a vacuum energy density, $\rho_v \sim (10^{-3}eV)^4$ today (in the cosmological units $\hbar = c = k = 1$). In a recent paper[41] we have argued that such a vacuum energy density is natural in the context of phase transitions linked to massive neutrinos. In fact, the neutrino masses required to solve the cosmological constant problem are consistent with those required to solve the solar neutrino problem by the MSW mechanism. We will return to a discussion of this subject in the last chapter. My hope is that this work will stimulate further thoughts about these issues.

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Figures for Chapter 2:

Figure 1: \mathcal{V} as a function for $\widehat{\mathcal{M}}^2$ for temperatures just above, at and just below the phase transition temperature.

Figure 2: $\mathcal{V}(\xi_1, \xi_3)$ for the temperatures used in fig. (2).

Chapter 3

REAL TIME APPLICATION: DISSIPATION VIA PARTICLE PRODUCTION IN SCALAR FIELD THEORIES

3.1 Introduction

In this chapter we will study the dynamics of dissipation in scalar field theories using the real time formalism. Dissipation and reheating are among the most important non-equilibrium phenomena that accompany any phase transition. In cosmology these process are particularly important in the context of inflation[1, 2]. Recall that at the end of new or chaotic inflation[3], defined by when the slow-roll conditions for the so-called inflaton field[4, 5], $\phi(t)$, fail to obtain, the inflaton begins to oscillate about its true ground state.

The dissipation of the coherent oscillations of the field due to particle production transfers the enormous vacuum energy of the fields prior to the phase transition into the energy of the particles. The subsequent thermalisation of these particles allows the inflationary cosmology to smoothly merge into the standard hot big bang cosmology. Futher, the enormous creation of entropy which accompanies the

particle production helps inflation solve the problems of standard cosmology.

In all realistic models of inflation, the inflaton is coupled to both fermionic and bosonic lighter fields. As the inflaton oscillates about the minimum of its potential, it decays to the lighter fields it is coupled to. This process is usually modeled by taking the evolution equation of the inflaton ϕ and inserting by hand a term which looks like $\Gamma\dot{\phi}$ [6, 7]. It is important to state that the $\Gamma\dot{\phi}$ term is not derivable starting from the fundamental Lagrangian or Hamiltonian of the system but has to be put in by hand just to model the dissipative dynamics. A more careful treatment is both desirable and required and this has been pointed also by Linde et al and Brandenberger et al[8]

This chapter will lay out the formalism that allows us to study the detailed dynamics of dissipation via particle production starting from the exact microscopic Hamiltonian of the system.

Though we have studied the time evolution of a scalar field coupled to fermions as well as another scalar field, in this chapter we will study the dynamics of a self-interacting scalar field. This is a natural first step. It will allow us to lay out the formalism in the simplest possible setting and allow us to gain insight into some of the essential physics in this simple setting.

One of the issues which frequently arises in trying to understand dissipation from a microscopic perspective is that dissipation appears to be an irreversible process whereas the underlying microscopic evolution equation are time reversible. In this context, it is important to realize the importance of two important factors that allow us to obtain dissipative dynamics from time reversal invariant evolution equations. These two factors are non-equilibrium initial conditions and coarse graining. Thus, even though the evolution equations are time reversal invariant, the non-equilibrium initial conditions make the behaviour of the system irreversible.

Further, coarse graining implies that even though total energy is conserved there is a transfer of energy from the modes of interest into the remaining modes which now play the role of an environment to the field modes or interest are coupled to.

Here is the plan for this chapter. For a self-interacting scalar field theory we study the time evolution of system by having the expectation value of the field away from the minimum of its potential. This provides the necessary non-equilibrium initial condition. The expectation value of the field starts oscillating about the minimum. These oscillations of the expectation value start damping out as the fluctuations start building up. We also study particle production taking place in the system. Thus, we arrive at the conclusion that there is a precise correlation between the damping out of the oscillations of the zero mode with the building up of the fluctuations and particle production.

We study two different self-interacting scalar field theories. One possessing a discrete symmetry and another possessing a continuous symmetry. For both these systems we investigate what happens when the symmetry is unbroken and then when the symmetry is broken. We will notice that the most dramatic damping takes place in the broken continuous symmetry case because of the availability of the Goldstone channel for dissipation.

We start off by setting up the formalism to study the time evolution. The time evolution of the system is determined by the quantum Liouville equation. We are interested in the time evolution of the expectation value of the field. (This time evolution equation is in fact a functional form of the Ehrenfest Theorem.) The field can be decomposed into the expectation value and fluctuations about the expectation value. As a first step we can try to do a loop expansion. However, we will notice that very soon the one-loop contribution becomes of the same order as the zeroth order contribution and hence one cannot trust the perturba-

tive expansion beyond this time. Thus, one is driven to consider ways to extract non-perturbative information. We next use the self-consistent Hartree Approximation to extract information of non-perturbation dynamics. We will then introduce the formalism to study particle production. The evolution equation we arrive at are coupled integro-differential equation which we solve numerically. Plots of the time evolution of the expectation value of the field, the growth of fluctuations and particle production will allow us to gain insight into the physics of this problems.

3.2 Non-equilibrium field theory and equations of motion

The generalization of statistical mechanics techniques to the description of non-equilibrium processes in quantum field theory has been available for a long time[11, 12, 13, 14, 15] but somehow has not yet been accepted as an integral part of the available tools to study field theory in extreme environments. We thus begin by presenting a somewhat pedagogical introduction to the subject for the non-practitioner. The discussion that follows is based on understanding gained through collaborations with Boyanovsky, de Vega, Holman and Lee[16, 17, 18, 19, 20]

The non-equilibrium description of a system is determined by the time evolution of the density matrix that describes it. This time evolution (in the Schrödinger picture) is determined by the quantum Liouville equation. A non-equilibrium situation arises whenever the Hamiltonian does not commute with the density matrix. Here we allow for an *explicitly* time dependent Hamiltonian, which might be the case if the system is in an external time dependent background, for example.

Our starting point is the Liouville equation for the density matrix of the system:

$$i\hbar \frac{\partial \rho(t)}{\partial t} = [H_{\text{evol}}, \rho(t)], \quad (3.1)$$

where H_{evol} is the Hamiltonian of the system that drives the non-equilibrium evolution of the system.

Next we use the density matrix to extract the order parameter from the field in the Schrödinger picture:

$$\phi(t) = \frac{1}{\Omega} \int d^3x \text{Tr} [\rho(t) \Phi(\vec{x})], \quad (3.2)$$

with Ω the spatial volume (later taken to infinity), and $\Phi(\vec{x})$ the field in the Schrödinger picture. Using the Liouville equation together with the Hamiltonian:

$$H = \int d^3x \left\{ \frac{\Pi^2}{2} + \frac{1}{2}(\nabla\Phi)^2 + V(\Phi) \right\} \quad (3.3)$$

and the standard equal time commutation relations between a field and its canonically conjugate momentum, we find the equations:

$$\frac{d\phi(t)}{dt} = \frac{1}{\Omega} \int d^3x \langle \Pi(\vec{x}, t) \rangle = \frac{1}{\Omega} \int d^3x \text{Tr} [\rho(t) \Pi(\vec{x})] = \pi(t) \quad (3.4)$$

$$\frac{d\pi(t)}{dt} = -\frac{1}{\Omega} \int d^3x \left\langle \frac{\delta V(\Phi)}{\delta \Phi(\vec{x})} \right\rangle. \quad (3.5)$$

From these equations we can find the equation of motion for the order parameter $\phi(t)$:

$$\frac{d^2\phi(t)}{dt^2} + \frac{1}{\Omega} \int d^3x \left\langle \frac{\delta V(\Phi)}{\delta \Phi(\vec{x})} \right\rangle = 0 \quad (3.6)$$

We expand the field operator as $\Phi(\vec{x}) = \phi(t) + \psi(\vec{x}, t)$, insert this expansion into equation (3.6) and keep only the quadratic terms in the fluctuation field $\psi(\vec{x}, t)$. Doing this yields the equation:

$$\frac{d^2\phi(t)}{dt^2} + V'(\phi(t)) + \frac{V'''(\phi(t))}{2\Omega} \int d^3x \langle \psi^2(\vec{x}, t) \rangle + \dots = 0 \quad (3.7)$$

Here the primes stand for derivatives with respect to ϕ .

At this point we need to compute and keep track of the fluctuations in the field. The fluctuations $\langle \psi^2(\vec{x}, t) \rangle$ is the two-point correlation function. In the language of field theory this is the two-point equal time Green's function. This Green's function can be evaluated in terms of the homogeneous solutions of the quadratic form for the fluctuations. For concreteness let us consider a self-interacting scalar field with the potential,

$$V(\Phi) = \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{4!}\Phi^4 \quad (3.8)$$

In this case the homogeneous solutions of the quadratic form for the fluctuations are determined by

$$\left[\frac{d^2}{dt^2} + \vec{k}^2 + m^2 + \frac{\lambda}{2} \phi^2(t) \right] U_k^\pm(t) = 0 \quad (3.9)$$

$$U_k^\pm(0) = 1 \ ; \ \dot{U}_k^\pm(0) = \mp i\omega_k^0 \quad (3.10)$$

with

$$\omega_k^0 = \left[\vec{k}^2 + m^2 + \frac{\lambda}{2} \phi^2(0) \right]^{\frac{1}{2}}. \quad (3.11)$$

The boundary conditions (3.10) correspond to positive U^+ and negative U^- frequency modes for $t < 0$ (the Wronskian of these solutions is $2i\omega_k^0$). Notice that $U_k^-(t) = [U_k^+(t)]^*$. In terms of these mode functions we obtain the two-point equal time Green's function to be

$$G(t, t) = \frac{i}{2\omega_k^0} U_k^+(t) U_k^-(t) \quad (3.12)$$

Thus to order \hbar we find the following equations

$$\ddot{\phi}(t) + m^2 \phi(t) + \frac{\lambda}{6} \phi^3(t) + \frac{\lambda \hbar}{2} \phi(t) \int \frac{d^3 k}{(2\pi)^3} \frac{|U_k^+(t)|^2}{2\omega_k^0} = 0 \quad (3.13)$$

$$\left[\frac{d^2}{dt^2} + \vec{k}^2 + m^2 + \frac{\lambda}{2} \phi^2(t) \right] U_k^+(t) = 0 \quad (3.14)$$

$$U_k^+(0) = 1 \ ; \ \dot{U}_k^+(0) = -i\omega_k^0 \quad (3.15)$$

where we have restored the \hbar to make the quantum corrections explicit. This set of equations clearly shows how the expectation value (coarse grained variable) “transfers energy” to the mode functions via a time dependent frequency, which then in turn modify the equations of motion for the expectation value. The equation for the mode functions, (3.14) may be solved in a perturbative expansion in terms of $\lambda \phi^2(t)$ involving the *retarded* Green's function.

Before attempting a numerical solution of the above equations it is important to understand the renormalization aspects. For this we need the large k behaviour

of the mode functions which is obtained via a WKB expansion as in references [22, 19] and to which the reader is referred to for details. We obtain

$$\int \frac{d^3k}{(2\pi)^3} \frac{|U_k^+(t)|^2}{2\omega_k^0} = \frac{\Lambda^2}{8\pi} - \frac{1}{8\pi} \left[m^2 + \frac{\lambda}{2} \phi^2(t) \right] \ln \left[\frac{\Lambda}{\kappa} \right] + \text{finite} \quad (3.16)$$

where Λ is an ultraviolet cutoff and κ an arbitrary renormalization scale. From the above expression it is clear how the mass and coupling constant are renormalized. It proves more convenient to subtract the one-loop contribution at $t = 0$ and absorb a finite renormalization in the mass, finally obtaining the renormalized equation of motion

$$\begin{aligned} \ddot{\phi}(t) + m_R^2 \phi(t) + \frac{\lambda_R}{6} \phi^3(t) + \frac{\lambda_R \hbar}{8\pi^2} \phi(t) \int_0^\Lambda k^2 dk \frac{[|U_k^+(t)|^2 - 1]}{\omega_k^0} \\ + \frac{\lambda_R^2 \hbar}{32\pi^2} \phi(t) (\phi^2(t) - \phi^2(0)) \ln [\Lambda/\kappa] = 0 \end{aligned} \quad (3.17)$$

In the equations for the mode functions the mass and coupling may be replaced by the renormalized quantities to this order. One would be tempted to pursue a numerical solution of these coupled equations. However doing so would not be consistent, since these equations were obtained only to order \hbar and a naive numerical solution will produce higher powers of \hbar that are not justified.

Within the spirit of the loop expansion we must be consistent and only keep terms of order \hbar . First we introduce dimensionless variables

$$\eta(t) = \sqrt{\frac{\lambda_R}{6m_R^2}} \phi(\tau) \quad ; \quad \tau = m_R t \quad ; \quad q = \frac{k}{m_R} \quad ; \quad g = \frac{\lambda_R \hbar}{8\pi^2} \quad (3.18)$$

and expand the field in terms of g as

$$\eta(\tau) = \eta_{cl}(\tau) + g\eta_1(\tau) + \dots \quad (3.19)$$

Now the equations of motion consistent up to $\mathcal{O}(\hbar)$ become

$$\ddot{\eta}_{cl}(\tau) + \eta_{cl}(\tau) + \eta_{cl}^3(\tau) = 0 \quad (3.20)$$

$$\ddot{\eta}_1(\tau) + \eta_1(\tau) + 3\eta_{cl}^2(\tau)\eta_1(\tau) + \eta_{cl}(\tau) \int_0^{\Lambda/m_R} q^2 dq \frac{[|U_q^+(\tau)|^2 - 1]}{[q^2 + 1 + 3\eta_{cl}^2(0)]^{1/2}} + \frac{3}{2}\eta_{cl}(\tau) (\eta_{cl}^2(\tau) - \eta_{cl}^2(0)) \ln[\Lambda/m_R] = 0 \quad (3.21)$$

The solution to equation (3.20) is an elliptic function. The equations for the mode functions become

$$\left[\frac{d^2}{d\tau^2} + q^2 + 1 + 3\eta_{cl}^2(\tau) \right] U_q^+(\tau) = 0 \quad (3.22)$$

with the boundary conditions as in eq.(3.15) in terms of the dimensionless frequencies and where for simplicity, we have chosen the renormalization scale $\kappa = m_R$. The chosen boundary conditions $\eta(0) = \eta_0$; $\dot{\eta}(0) = 0$ can be implemented as

$$\eta_{cl}(0) = \eta_0 \ ; \ \dot{\eta}_{cl}(0) = 0 \ ; \ \eta_1(0) = 0 \ ; \ \dot{\eta}_1(0) = 0 \quad (3.23)$$

In fig.(2) we show $\eta_1(\tau)$ with the above boundary conditions with $\eta_0 = 1$ and $g = 0.1$. A cutoff $\Lambda/m_R = 100$ was chosen but no cut-off sensitivity was detected by varying the cutoff by a factor 3. Notice that the amplitude grows as a function of time. This is the behavior shown in fig.(2).

Failure of perturbation theory to describe dissipation

This section has been devoted to a perturbative analysis of the “dissipative aspects” of the equation of motion for the scalar field. Perturbation theory has been carried out as an expansion up to $\mathcal{O}(\hbar)$, both in the broken and the unbroken symmetry case. The potential for the unbroken and broken symmetry cases is given in figures 3 and 4 respectively. In both cases we found that the amplitude of the quantum corrections *grow as a function of time* and that the long time behavior cannot be captured in perturbation theory. This failure of perturbation theory to describe dissipation is clearly understood from a very elementary but yet illuminating example: the damped harmonic oscillator. Consider a damped harmonic

oscillator

$$\ddot{q} + \Gamma \dot{q} + q = 0 \quad (3.24)$$

with $\Gamma \sim \mathcal{O}(\lambda)$ where λ is a small perturbative coupling. The above equation can be solved exactly to yield the solution,

$$q(t) = e^{-\frac{\Gamma}{2}t} \cos[\omega(\Gamma^2)t] \approx \cos(t) - \frac{\Gamma}{2}t \cos(t) + \mathcal{O}(\Gamma^2). \quad (3.25)$$

We see that there does exist a perturbative expansion in Γ , but in order to find appreciable damping, we must wait a time $\sim \mathcal{O}(1/\Gamma)$ at which perturbation theory becomes unreliable.

In order to properly describe dissipation and damping one must resum the perturbative expansion. Another hint that points to a resummation of the perturbative series is provided by the set of equations eqs.(3.20-3.22). In eq.(3.20), the classical solution is a periodic function of time of constant amplitude, since the classical equation has a conserved energy. As a consequence, the “potential” in the equation for the modes (eq.(3.22)) is a periodic function of time with constant amplitude. Thus although the fluctuations react back on the coarse grained field, only the classically conserved part of the motion of the coarse grained field enters in the evolution equations of the mode functions. This is a result of being consistent with the loop expansion, but clearly this approximation is not energy conserving.

As we will point out in the next section, in an energy conserving scheme the fluctuations and amplitudes will grow up to a maximum value and then will always remain bounded at all times.

Thus in summary for this section, we draw the conclusion that perturbation theory is not sufficient (without major ad-hoc assumptions) to capture the physics of dissipation and damping in real time. A resummation scheme is needed that

effectively sums up the whole (or partial) perturbative series in a consistent and/or controlled manner, and which provides a reliable estimate for the long-time behavior. The next section is devoted to the analytical and numerical study of some of these schemes.

3.3 Non-perturbative schemes I: Hartree approximation

Motivated by the failure of the loop expansion, we now proceed to consider the equations of motion in some non-perturbative schemes. First we study a single scalar model in the time dependent Hartree approximation. After this, we study an $O(N)$ scalar theory in the large N limit. This last case allows us to study the effect of Goldstone bosons on the time evolution of the order parameter.

In a single scalar model with the interactions described by the potential of eq.(3.8), the Hartree approximation is implemented as follows. We again decompose the fields as $\Phi = \phi + \psi$. The Hartree approximation is implemented by factorising all powers of the field higher than two and expressing them in terms of products of two point correlation functions and upto quadratic terms in the field and demanding that a self-consistency condition be satisfied. Thus, the Hartree approximation is obtained by assuming the factorization

$$\begin{aligned}\psi^3(\vec{x}, t) &\rightarrow 3\langle\psi^2(\vec{x}, t)\rangle\psi(\vec{x}, t) \\ \psi^4(\vec{x}, t) &\rightarrow 6\langle\psi^2(\vec{x}, t)\rangle\psi^2(\vec{x}, t) - 3\langle\psi^2(\vec{x}, t)\rangle^2\end{aligned}\tag{3.26}$$

Translational invariance shows that $\langle\psi^2(\vec{x}, t)\rangle$ can only be a function of time. The expectation value will be determined within a self-consistent approximation.

In this approximation the resulting Hamiltonian is quadratic, with a linear term in ψ :

$$H_H(t) = \int d^3x \left\{ \frac{\Pi^2}{2} + \frac{(\nabla\psi)^2}{2} + \psi\mathcal{V}^1(t) + \frac{\mathcal{M}^2(t)}{2}\psi^2 \right\}. \quad (3.27)$$

Here Π is the canonical momenta conjugate to $\psi(\vec{x})$ and \mathcal{V}^1 is given by,

$$\mathcal{V}^{(1)}(t) = V'(\phi) + \frac{1}{2}\lambda\phi\langle\psi^2\rangle \quad (3.28)$$

It is recognized as the derivative of the Hartree “effective potential”[23, 24] with respect to ϕ (it is the derivative of the non-gradient terms of the effective action[16, 18, 25]).

Also,

$$\mathcal{M}^2(t) = V''(\phi) + \frac{\lambda}{2}\langle\psi^2(t)\rangle = m^2 + \frac{\lambda}{2}\phi^2(t) + \frac{\lambda}{2}\langle\psi^2(t)\rangle \quad (3.29)$$

With this Hartree factorization we can use the non-equilibrium formalism outlined in the previous section to determine the time evolution. The resulting equations are obtained to be:

$$\ddot{\phi} + m^2\phi + \frac{\lambda}{6}\phi^3 + \frac{\lambda}{2}\phi\langle\psi^2(t)\rangle = 0 \quad (3.30)$$

$$\langle\psi^2(t)\rangle = \int \frac{d^3k}{(2\pi)^3} [-iG_k(t, t)] = \int \frac{d^3k}{(2\pi)^3} \frac{|U_k^+(t)|^2}{2\omega_k(0)} \quad (3.31)$$

$$\left[\frac{d^2}{dt^2} + \omega_k^2(t) \right] U_k^+(t) = 0 \quad ; \quad \omega_k^2(t) = \vec{k}^2 + \mathcal{M}^2(t) \quad (3.32)$$

The initial conditions for the mode functions are

$$U_k^+(0) = 1 \quad ; \quad \dot{U}_k^+(0) = -i\omega_k(0) \quad (3.33)$$

It is clear that the Hartree approximation makes the Lagrangian density quadratic at the expense of a self-consistent condition. In the time independent case, this approximation sums up all the “daisy” (or “cactus”) diagrams and leads to a self-consistent gap equation.

At this stage, we must point out that the Hartree approximation is uncontrolled in this single scalar theory. This approximation does, however, become exact in the $N \rightarrow \infty$ limit of the $O(N)$ model which we will discuss in the next section.

3.3.1 Renormalization

We now analyze the renormalization aspects within the Hartree approximation. To study the renormalization we need to understand the divergences in this integral

$$\langle \psi^2(t) \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{|U_k^+(t)|^2}{2\omega_k(0)} \quad (3.34)$$

The divergences will be determined from the large- k behavior of the mode functions which obey the differential equations obtained from (3.32) with the initial conditions (3.33). By a WKB-type analysis (see[22, 19] for a detailed description), in the $k \rightarrow \infty$ limit, we find

$$\frac{|U_k^+(t)|^2}{2\omega_k(0)} = \frac{1}{k} - \frac{1}{4k^3} \left[m^2 + \frac{\lambda}{2} \phi^2(t) + \frac{\lambda}{2} \langle \psi^2(t) \rangle \right] + \mathcal{O}\left(\frac{1}{k^4}\right) + \dots \quad (3.35)$$

Inserting these results in the integral, it is straightforward to find the divergent terms and we find

$$\int \frac{d^3k}{(2\pi)^3} \frac{|U_k^+(t)|^2}{2\omega_k(0)} = \frac{1}{8\pi^2} \Lambda^2 - \frac{1}{8\pi^2} \ln\left(\frac{\Lambda}{\kappa}\right) \left[m^2 + \frac{\lambda}{2} \phi^2(t) + \frac{\lambda}{2} \langle \psi^2(t) \rangle \right] + \text{finite} \quad (3.36)$$

where Λ is an upper momentum cutoff and κ a renormalization scale.

Now, we are in position to specify the renormalization prescription within the Hartree approximation. In this approximation, there are no interactions, since the Lagrangian density is quadratic. The nonlinearities are encoded in the self-consistency conditions. Because of this, there are no counterterms with which to cancel the divergence and the differential equation for the mode functions must be

finite. Therefore, it leads to the following renormalization prescription:

$$m_B^2 + \frac{\lambda_B}{2}\phi^2(t) + \frac{\lambda_B}{2}\langle\psi^2(t)\rangle_B = m_R^2 + \frac{\lambda_R}{2}\phi^2(t) + \frac{\lambda_R}{2}\langle\psi^2(t)\rangle_R \quad (3.37)$$

where the subscripts B, R refer to the bare and renormalized quantities respectively and $\langle\psi^2(t)\rangle_B$ is read from eq.(3.36):

$$\langle\psi^2(t)\rangle_B = \frac{1}{8\pi^2}\Lambda^2 - \frac{1}{8\pi^2}\ln\left(\frac{\Lambda}{\kappa}\right) \left[m_R^2 + \frac{\lambda_R}{2}\phi^2(t) + \frac{\lambda_R}{2}\langle\psi^2(t)\rangle_R \right] + \text{finite} \quad (3.38)$$

Using this renormalization prescription eq.(3.37), we obtain

$$m_B^2 + \frac{\lambda_B}{16\pi^2}\Lambda^2 = m_R^2 \left[1 + \frac{\lambda_R}{16\pi^2}\ln\left(\frac{\Lambda}{\kappa}\right) \right] \quad (3.39)$$

$$\lambda_B = \frac{\lambda_R}{1 - \frac{\lambda_R}{16\pi^2}\ln\left(\frac{\Lambda}{\kappa}\right)} \quad (3.40)$$

and

$$\langle\psi^2(t)\rangle_R = \left[\frac{1}{1 - \frac{\lambda_R}{16\pi^2}\ln\left(\frac{\Lambda}{\kappa}\right)} \right] \times \left\{ \int \frac{d^3k}{(2\pi)^3} \frac{|U_k^+(t)|^2}{2\omega_k(0)} - \left[\frac{1}{8\pi^2}\Lambda^2 - \frac{1}{8\pi^2}\ln\left(\frac{\Lambda}{\kappa}\right) \left(m_R^2 + \frac{\lambda_R}{2}\phi^2(t) \right) \right] \right\}$$

It is clear that there is no wavefunction renormalization. This is a consequence of the approximation invoked. There is, in fact, no wavefunction renormalization in either one-loop or Hartree approximation for a scalar field theory in three spatial dimensions.

With an eye towards the numerical analysis, it is more convenient to write

$$\langle\psi^2(t)\rangle_R = \left(\langle\psi^2(t)\rangle_R - \langle\psi^2(0)\rangle_R \right) + \langle\psi^2(0)\rangle_R \quad (3.41)$$

and perform a subtraction at time $t = 0$ absorbing $\langle\psi^2(0)\rangle_R$ into a further *finite* renormalization of the mass term ($m_R^2 + \langle\psi^2(0)\rangle_R = M_R^2$).

The renormalized equations that we will solve finally become

$$\ddot{\phi} + M_R^2 \phi + \frac{\lambda_R}{2} \left[1 - \left(\frac{2}{3} \right) \frac{1}{1 - \frac{\lambda_R}{16\pi^2} \ln \left(\frac{\Lambda}{\kappa} \right)} \right] \phi^3 + \frac{\lambda_R}{2} \phi \left(\langle \psi^2(t) \rangle_R - \langle \psi^2(0) \rangle_R \right) = 0 \quad (3.42)$$

$$\left[\frac{d^2}{dt^2} + k^2 + M_R^2 + \frac{\lambda_R}{2} \phi^2(t) + \frac{\lambda_R}{2} \left(\langle \psi^2(t) \rangle_R - \langle \psi^2(0) \rangle_R \right) \right] U_k^+(t) = 0 \quad (3.43)$$

and

$$\begin{aligned} \left(\langle \psi^2(t) \rangle_R - \langle \psi^2(0) \rangle_R \right) &= \left[\frac{1}{1 - \frac{\lambda_R}{16\pi^2} \ln \left(\frac{\Lambda}{\kappa} \right)} \right] \times \\ &\quad \left\{ \int \frac{d^3 k}{(2\pi)^3} \frac{|U_k^+(t)|^2 - 1}{2\omega_k(0)} + \frac{\lambda_R}{16\pi^2} \ln \left(\frac{\Lambda}{\kappa} \right) \left(\phi^2(t) - \phi^2(0) \right) \right\} \end{aligned}$$

with the initial conditons for $U_k^+(t)$:

$$U_k^+(0) = 1 \quad ; \quad \dot{U}_k^+(0) = -i\omega_k(0) \quad ; \quad \omega_k(0) = \sqrt{k^2 + M_R^2 + \frac{\lambda_R}{2} \phi^2(0)} \quad (3.44)$$

It is worth noticing that there is a weak cutoff dependence on the renormalized equations of motion of the order parameter and the mode functions. This is a consequence of the well known “triviality” problem of the scalar quartic interaction in four space-time dimensions. This has the consequence that for a fixed renormalized coupling the cutoff must be kept fixed and finite. The presence of the Landau pole prevents taking the limit of the ultraviolet cutoff to infinity while keeping the renormalized coupling fixed.

This theory is sensible only as a low-energy cutoff effective theory. We then must be careful that for a fixed value of λ_R , the cutoff must be such that the theory never crosses the Landau pole. Thus from a numerical perspective there will always be a cutoff sensitivity in the theory. However, for small coupling we expect the cutoff dependence to be rather weak (this will be confirmed numerically) provided the cutoff is far away from the Landau pole.

3.3.2 Particle Production

Before we engage ourselves in a numerical integration of the above equations of motion we want to address the issue of particle production since it is of great importance for the understanding of dissipative processes.

In what follows, we consider particle production due to the time varying effective mass $\mathcal{M}^2(t)$ in eq.(3.29) of the quantum field ψ for the single scalar model.

In a time dependent background the concept of particle is ambiguous and it must be defined with respect to some particular state. Let us consider the Heisenberg fields at $t = 0$ written as

$$\begin{aligned}\psi(\vec{x}, 0) &= \frac{1}{\sqrt{\Omega}} \sum_k \frac{1}{\sqrt{2\omega_k(0)}} \left(a_k(0) + a_{-k}^\dagger(0) \right) e^{i\vec{k} \cdot \vec{x}} \\ \Pi_\psi(\vec{x}, 0) &= \frac{-i}{\sqrt{\Omega}} \sum_k \sqrt{\frac{\omega_k(0)}{2}} \left(a_k(0) - a_{-k}^\dagger(0) \right) e^{i\vec{k} \cdot \vec{x}}\end{aligned}\quad (3.45)$$

with $\omega_k(0)$ as in eq.(3.32) and Ω the spatial volume. The Hamiltonian is diagonalized at $t = 0$ by these creation and destruction operators:

$$H(0) = \sum_k \omega_k(0) \left[a_k^\dagger(0) a_k(0) + \frac{1}{2} \right] \quad (3.46)$$

Thus we define the Hartree-Fock states at $t = 0$ as the vacuum annihilated by $a_k(0)$ together with the tower of excitations obtained by applying polynomials in $a_k^\dagger(0)$ to this vacuum state. The Hartree-Fock vacuum state at $t = 0$ is chosen as the reference state. As time passes, particles (as defined with respect to this state) will be produced as a result of parametric amplification[26, 27]. We should mention that our definition differs from that of other authors[21, 27] in that we chose the state at time $t = 0$ rather than using the adiabatic modes (that diagonalize the instantaneous Hamiltonian).

We define the number density of particles as a function of time as

$$\mathcal{N}(t) = \int \frac{d^3k}{(2\pi)^3} \frac{\text{Tr} [a_k^\dagger(0)a_k(0)\rho(t)]}{\text{Tr}\rho(0)} = \int \frac{d^3k}{(2\pi)^3} \frac{\text{Tr} [a_k^\dagger(t)a_k(t)\rho(0)]}{\text{Tr}\rho(0)} \quad (3.47)$$

where by definition

$$a_k^\dagger(t) = U^{-1}(t, 0)a_k^\dagger(0)U(t, 0) ; a_k(t) = U^{-1}(t, 0)a_k(0)U(t, 0) \quad (3.48)$$

are the time-evolved operators in the Heisenberg picture. In terms of these time evolved operators, we may write:

$$\begin{aligned} \psi(\vec{x}, t) &= \frac{1}{\sqrt{\Omega}} \sum_k \frac{1}{\sqrt{2\omega_k(0)}} (a_k(t) + a_{-k}^\dagger(t)) e^{i\vec{k}\cdot\vec{x}} \\ \Pi_\psi(\vec{x}, t) &= \frac{-i}{\sqrt{\Omega}} \sum_k \sqrt{\frac{\omega_k(0)}{2}} (a_k(t) - a_{-k}^\dagger(t)) e^{i\vec{k}\cdot\vec{x}} \end{aligned} \quad (3.49)$$

On the other hand, we now expand the Heisenberg fields at time t in the following orthonormal basis

$$\begin{aligned} \psi(\vec{x}, t) &= \frac{1}{\sqrt{\Omega}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k(0)}} (\tilde{a}_k U_k^+(t) + \tilde{a}_{-k}^\dagger U_k^-(t)) e^{i\vec{k}\cdot\vec{x}} \\ \Pi_\psi(\vec{x}, t) &= \frac{1}{\sqrt{\Omega}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k(0)}} (\tilde{a}_k \dot{U}_k^+(t) + \tilde{a}_{-k}^\dagger \dot{U}_k^-(t)) e^{i\vec{k}\cdot\vec{x}} \end{aligned} \quad (3.50)$$

where the mode functions $U_k^+(t)$; $U_k^-(t) = (U_k^+(t))^*$ are the Hartree-Fock mode functions obeying eqs.(3.43-3.44) together with the self consistency condition.

Thus $\tilde{a}_k, \tilde{a}_k^\dagger$ are the annihilation and creation operators of Hartree-Fock states, and the Heisenberg field $\psi(\vec{x}, t)$ is a solution of the Heisenberg equations of motion in the Hartree approximation. Therefore the \tilde{a}_k^\dagger and \tilde{a}_k do not depend on time and are identical to $a_k^\dagger(0)$ and $a_k(0)$ respectively (we can check this by evaluating the expansion in eq.(3.50) at $t = 0$ together with the initial conditions on $U_k^+(t)$ in eq.(3.33)). Using the Wronskian properties of the function $U_k^+(t)$, we see that the \tilde{a}_k^\dagger and \tilde{a}_k satisfy the usual canonical commutation relations. The reason for the

choice of the vacuum state at $t = 0$ now becomes clear; this is the initial time at which the boundary conditions on the modes are determined. The mode functions $U_k^+(t)$; $U_k^-(t)$ are then identified with positive and negative frequency modes at the initial time.

By comparing the expansion in eq.(3.45) evaluated at time t with that in eq.(3.50), we find that the creation and annihilation operator at time t can be related to those at time $t = 0$ via a Bogoliubov transformation:

$$a_k(t) = \mathcal{F}_{+,k}(t)a_k(0) + \mathcal{F}_{-,k}(t)a_{-k}^\dagger(0) \quad (3.51)$$

The $\mathcal{F}_\pm(t)$ can be read off in terms of the mode functions $U_k^+(t)$

$$\begin{aligned} |\mathcal{F}_{+,k}(t)|^2 &= \frac{1}{4} |U_k^+(t)|^2 \left[1 + \frac{|\dot{U}_k^+(t)|^2}{\omega_k^2(0) |U_k^+(t)|^2} \right] + \frac{1}{2} \\ |\mathcal{F}_{+,k}(t)|^2 - |\mathcal{F}_{-,k}(t)|^2 &= 1 \end{aligned} \quad (3.52)$$

At any time t the expectation value of the number operator for the quanta of ψ in each k -mode is given by

$$\mathcal{N}_k(t) = \frac{Tr \left[a_k^\dagger(t) a_k(t) \rho(0) \right]}{Tr \rho(0)} \quad (3.53)$$

After some algebra, we find

$$\mathcal{N}_k(t) = \left(2 |\mathcal{F}_{+,k}(t)|^2 - 1 \right) \mathcal{N}_k(0) + \left(|\mathcal{F}_{+,k}(t)|^2 - 1 \right) \quad (3.54)$$

This result exhibits the contributions from “spontaneous” (proportional to the initial particle occupations) and “induced” (independent of it) particle production. Since we are analyzing the zero temperature case with $\mathcal{N}_k(0) = 0$ only the induced contribution results.

3.3.3 Numerical Analysis

Unbroken Symmetry Case

In order to perform a numerical analysis it is necessary to introduce dimensionless quantities and it becomes convenient to chose the renormalization point $\kappa = M_R$.

Thus we define

$$\begin{aligned}\eta(t) &= \phi(t)\sqrt{\frac{\lambda_R}{2M_R^2}}; \quad q = \frac{k}{M_R}; \quad \tau = M_R t; \quad g = \frac{\lambda_R}{8\pi^2} \\ \Sigma(t) &= \frac{4\pi^2}{M_R^2} (\langle \psi^2(t) \rangle_R - \langle \psi^2(0) \rangle_R)\end{aligned}\tag{3.55}$$

and finally, the equations of motion become

$$\begin{aligned}\frac{d^2}{d\tau^2}\eta + \eta + \left[1 - \left(\frac{2}{3}\right) \frac{1}{1 - \frac{g}{2} \ln\left(\frac{\Lambda}{M_R}\right)}\right] \eta^3 + g\eta\Sigma(\tau) &= 0 \\ \left[\frac{d^2}{d\tau^2} + q^2 + 1 + \eta^2(\tau) + g\Sigma(\tau)\right] U_q^+(\tau) &= 0\end{aligned}\tag{3.56}$$

$$U_q^+(0) = 1; \quad \frac{d}{d\tau}U_q^+(0) = -i\sqrt{q^2 + 1 + \eta^2(0)}\tag{3.57}$$

$$\begin{aligned}\Sigma(\tau) &= \left[\frac{1}{1 - \frac{g}{2} \ln\left(\frac{\Lambda}{M_R}\right)}\right] \times \\ &\left\{ \int_0^{\Lambda/M_R} q^2 dq \frac{|U_q^+(\tau)|^2 - 1}{\sqrt{q^2 + 1 + \eta^2(0)}} + \frac{1}{2} \ln\left(\frac{\Lambda}{M_R}\right) (\eta^2(\tau) - \eta^2(0)) \right\}\end{aligned}\tag{3.58}$$

In terms of the dimensionless quantities we obtain the number of particles within a *correlation volume* $N(\tau) = \mathcal{N}(t)/M_R^3$

$$N(\tau) = \frac{1}{8\pi^2} \int_0^{\Lambda/M_R} q^2 dq \left\{ |U_q^+(\tau)|^2 + \frac{|\dot{U}_q^+(\tau)|^2}{\sqrt{q^2 + 1 + \eta^2(0)}} - 2 \right\}\tag{3.59}$$

Figures (5.a,b,c) show $\eta(\tau)$, $\Sigma(\tau)$ and $N(\tau)$ in the Hartree approximation, for $g = 0.1$, $\eta(0) = 1.0$ and $\Lambda/M_R = 100$; we did not detect an appreciable cutoff dependence by varying the cutoff between 50 and 200. Clearly there is

no appreciable damping in $\eta(\tau)$. In fact it can be seen that the period of the oscillation is very close to 2π , which is the period of the classical solution of the linear theory. This is understood because the coefficient of the cubic term is very small and $g\Sigma(\tau)$ is negligible. Particle production is also negligible. This situation should be contrasted with that shown in figures (6.a-c) and (7.a-c) in which there is dissipation and damping in the evolution of $\eta(\tau)$ for $\eta(0) = 4, 5$ respectively and the same values for g and the cutoff. There are several noteworthy features that can be deduced from these figures. First the fluctuations become very large, such that $g\Sigma(\tau)$ becomes $\mathcal{O}(1)$. Second, figures (6.a) and (7.a) clearly show that initially, channels are open and energy is transferred away from the $q = 0$ mode of the field. Eventually however, these channels shut off, and the dynamics of the expectation value settles into an oscillatory motion. The time scale for the shutting off of the dissipative behavior decreases as $\eta(0)$ increases; it is about 25 for $\eta(0) = 4$, and about 18 for $\eta(0) = 5$. This time scale is correlated with the time scale in which particles are produced by parametric amplification and the quantum fluctuations begin to plateau (figures (6.b,c) and (7.b,c)). Clearly the dissipative mechanism which damps the motion of the expectation value is particle production. Furthermore, the long time dynamics for the expectation value *does not* correspond to exponential damping. In fact, we did not find any appreciable damping for $\tau \geq 70$ in these cases. It is illuminating to compare this situation with that of a smaller coupling depicted in figures (8.a-c) for $\eta(0) = 5$ $g = 0.05$. Clearly the time scale for damping is much larger and there is still appreciable damping at $\tau \approx 100$. Notice also that $g\Sigma(\tau) \approx 2$ and that particles are being produced even at long times and this again correlates with the evidence that the field expectation value shows damped motion at long times, clearly showing that the numerical analysis has not reached the asymptotic regime.

The fundamental question to be raised at this point is: what is the origin of the damping in the evolution of the field expectation value? Clearly this is a collisionless process as collisions are not taken into account in the Hartree approximation (although the one-loop diagram that enters in the two particle collision amplitude with the two-particle cut is contained in the Hartree approximation and is responsible for thresholds to particle production). The physical mechanism is very similar to that of Landau damping in the collisionless Vlasov equation for plasmas[9] and also found in the study of strong electric fields in reference[10]. In the case under consideration, energy is transferred from the expectation value to the quantum fluctuations which back-react on the evolution of the field expectation value but out of phase. This phase difference between the oscillations of $\eta^2(\tau)$ and those of $\Sigma(\tau)$ can be clearly seen to be π in figures (6.a,b), (7.a,b) since the maxima of $\eta^2(\tau)$ occur at the same times as the minima of $\Sigma(\tau)$ and vice versa.

This is an important point learned from our analysis and that is not *a priori* taken into account in the usual arguments for dissipation via collisions. The process of thermalization, however, will necessarily involve collisions and cannot be studied within the schemes addressed in this paper.

Broken Symmetry

The broken symmetry case is obtained by writing $M_R^2 = -\mu_R^2 < 0$ and using the scale μ_R instead of M_R to define the dimensionless quantities as in eq.(3.55) and the renormalization scale. The equations of motion in this case become

$$\frac{d^2}{d\tau^2}\eta - \eta + \left[1 - \left(\frac{2}{3}\right) \frac{1}{1 - \frac{g}{2} \ln\left(\frac{\Lambda}{\mu_R}\right)}\right] \eta^3 + g\eta\Sigma(\tau) = 0$$

$$\left[\frac{d^2}{d\tau^2} + q^2 - 1 + \eta^2(\tau) + g\Sigma(\tau)\right] U_q^+(\tau) = 0 \quad (3.60)$$

$$U_q^+(0) = 1 ; \quad \frac{d}{d\tau}U_q^+(0) = -i\sqrt{q^2 + 1 + \eta^2(0)} \quad (3.61)$$

with $\Sigma(\tau)$ given in eq.(3.58) but with M_R replaced by μ_R . The broken symmetry case is more subtle because of the possibility of unstable modes for initial conditions in which $\eta(0) \ll 1$. We have kept the boundary conditions of eq.(3.61) the same as in eq.(3.57). This corresponds to preparing an initial state as a Gaussian state centered at $\eta(0)$ with real and positive covariance (width) and letting it evolve for $t > 0$ in the broken symmetry potential[22, 33]. The number of particles produced within a correlation volume (now μ_R^{-3}) is given by eq.(3.59) with M_R replaced by μ_R .

Figures (9.a-c) depict the dynamics for a broken symmetry case in which $\eta(0) = 10^{-5}$ i.e. very close to the top of the potential hill. Notice that as the field expectation value rolls down the hill the unstable modes make the fluctuation grow dramatically until about $\tau \approx 50$ at which point $g\Sigma(\tau) \approx 1$. At this point, the unstable growth of fluctuations shuts off and the field begins damped oscillatory motion around a mean value of about ≈ 1.2 . This point is a minimum of the effective action. The damping of these oscillations is very similar to the damping around the origin in the unbroken case. Most of the particle production and the largest quantum fluctuations occur when the field expectation value is rolling down the region for which there are unstable frequencies for the mode functions (see eq.(3.60)). This behavior is similar to that found previously by some of these authors[22].

3.4 Non-perturbative schemes II: Large N limit in the $O(N)$ Model

Although the Hartree approximation offers a non-perturbative resummation of select terms, it is not a consistent approximation because there is no *a priori* small parameter that defines the approximation. Furthermore we want to study the

effects of dissipation by Goldstone bosons in a non perturbative but controlled expansion.

In this section, we consider the $O(N)$ model in the large N limit. The large N limit has been used in studies of non-equilibrium dynamics[28, 29, 30, 33] and it provides a very powerful tool for studying non-equilibrium dynamics non-perturbatively in a consistent manner. The Lagrangian density is the following:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi}-V(\sigma,\vec{\pi}) \\ V(\sigma,\vec{\pi}) &= \frac{1}{2}m^2\vec{\phi}\cdot\vec{\phi}+\frac{\lambda}{8N}(\vec{\phi}\cdot\vec{\phi})^2\end{aligned}\tag{3.62}$$

for λ fixed in the large N limit. Here $\vec{\phi}$ is an $O(N)$ vector, $\vec{\phi} = (\sigma, \vec{\pi})$ and $\vec{\pi}$ represents the $N-1$ pions. In what follows, we will consider two different cases of the potential $V(\sigma, \vec{\pi})$, with ($m^2 < 0$) or without ($m^2 > 0$) symmetry breaking.

Our first order of business is to identify the correct order parameter for the phase transition and then to obtain its equation of motion. Let us define the fluctuation field operator $\chi(\vec{x}, t)$ as

$$\sigma = \phi_0(t) + \chi(\vec{x}, t),\tag{3.63}$$

with $\phi_0(t)$ a c-number field defined by:

$$\begin{aligned}\phi_0(t) &= \frac{1}{\Omega} \int d^3x \langle \sigma(\vec{x}) \rangle \\ &= \frac{1}{\Omega} \int d^3x \frac{\text{Tr}(\rho(t)\sigma(\vec{x}))}{\text{Tr}\rho(t)}.\end{aligned}\tag{3.64}$$

We can use the Liouville equation as before to arrive at the following evolution equation for the order parameter $\phi_0(t)$:

$$\ddot{\phi}_0(t) + \frac{1}{\Omega} \int d^3x \left\langle \frac{\delta V(\sigma, \vec{\pi})}{\delta \sigma(\vec{x})} \right\rangle = 0.\tag{3.65}$$

To proceed further we have to determine the density matrix. Since the Liouville equation is first order in time we need only specify $\rho(t = 0)$. At this stage we could proceed to a perturbative description of the dynamics (in a loop expansion).

However, as we learned previously in a similar situation[16, 17] and as we have argued in this chapter, the non-equilibrium dynamics of the phase transition cannot be studied within perturbation theory.

In the presence of a vacuum expectation value, the Hartree factorization is somewhat subtle. We will make a series of *assumptions* that we feel are quite reasonable but which, of course, may fail to hold under some circumstances and for which we do not have an *a priori* justification. These are the following: i) no cross correlations between the pions and the sigma field, and ii) that the two point correlation functions of the pions are diagonal in isospin space, where by isospin we now refer to the unbroken $O(N)$ symmetry under which the pions transform as a triplet. These assumptions lead to the following Hartree factorization of the non-linear terms in the Hamiltonian:

$$\chi^4 \rightarrow 6\langle\chi^2\rangle\chi^2 + \text{constant} \quad (3.66)$$

$$\chi^3 \rightarrow 3\langle\chi^2\rangle\chi \quad (3.67)$$

$$(\vec{\pi} \cdot \vec{\pi})^2 \rightarrow \left(2 + \frac{4}{N-1}\right) \langle\vec{\pi}^2\rangle\vec{\pi}^2 + \text{constant} \quad (3.68)$$

$$\vec{\pi}^2\chi^2 \rightarrow \langle\vec{\pi}^2\rangle\chi^2 + \vec{\pi}^2\langle\chi^2\rangle \quad (3.69)$$

$$\vec{\pi}^2\chi \rightarrow \langle\vec{\pi}^2\rangle\chi \quad (3.70)$$

where by “constant” we mean the operator independent expectation values of the composite operators which will not enter into the time evolution equation of the order parameter.

In this approximation the resulting Hamiltonian is quadratic, with a linear term

in χ :

$$H_H(t) = \int d^3x \left\{ \frac{\Pi_\chi^2}{2} + \frac{\vec{\Pi}_\pi^2}{2} + \frac{(\nabla\chi)^2}{2} + \frac{(\nabla\vec{\pi})^2}{2} + \chi\mathcal{V}^1(t) + \frac{\mathcal{M}_\chi^2(t)}{2}\chi^2 + \frac{\mathcal{M}_\pi^2(t)}{2}\vec{\pi}^2 \right\}. \quad (3.71)$$

Here Π_χ , $\vec{\Pi}_\pi$ are the canonical momenta conjugate to $\chi(\vec{x})$, $\vec{\pi}(\vec{x})$ respectively and \mathcal{V}^1 is recognized as the derivative of the Hartree “effective potential” [23, 24] with respect to ϕ_0 (it is the derivative of the non-gradient terms of the effective action [16, 18, 25]).

To obtain a large N limit, we define

$$\langle \vec{\pi}^2 \rangle = N \langle \psi^2 \rangle \quad (3.72)$$

$$\phi_0(t) = \phi(t) \sqrt{N} \quad (3.73)$$

with

$$\langle \psi^2 \rangle \approx \mathcal{O}(1) ; \langle \chi^2 \rangle \approx \mathcal{O}(1) ; \phi \approx \mathcal{O}(1). \quad (3.74)$$

We will approximate further by neglecting the $\mathcal{O}(\frac{1}{N})$ terms in the formal large N limit. We now obtain

$$V'(\phi(t), t) = \sqrt{N} \phi(t) \left[m^2 + \frac{\lambda}{2} \phi^2(t) + \frac{\lambda}{2} \langle \psi^2(t) \rangle \right] \quad (3.75)$$

$$\mathcal{M}_\pi^2(t) = m^2 + \frac{\lambda}{2} \phi^2(t) + \frac{\lambda}{2} \langle \psi^2(t) \rangle \quad (3.76)$$

$$\mathcal{M}_\chi^2(t) = m^2 + \frac{3\lambda}{2} \phi^2(t) + \frac{\lambda}{2} \langle \psi^2(t) \rangle \quad (3.77)$$

We obtain the following set of equations:

$$\begin{aligned} \ddot{\phi}(t) + \phi(t) \left[m^2 + \frac{\lambda}{2} \phi^2(t) + \frac{\lambda}{2} \langle \psi^2(t) \rangle \right] &= 0 \\ \langle \psi^2(t) \rangle &= \int \frac{d^3k}{(2\pi)^3} \frac{|U_k^+(t)|^2}{2\omega_{\vec{\pi}k}^2(0)} \end{aligned} \quad (3.78)$$

$$\left[\frac{d^2}{dt^2} + \omega_{\vec{\pi}k}^2(t) \right] U_k^+(t) = 0 ; \omega_{\vec{\pi}k}^2(t) = \vec{k}^2 + \mathcal{M}_\pi^2(t) \quad (3.79)$$

The initial conditions for the mode functions $U_k^+(t)$ are

$$U_k^+(0) = 1 \quad ; \quad \dot{U}_k^+(0) = -i\omega_{\vec{\pi}k}(0) \quad (3.80)$$

Since in this approximation, the dynamics for the $\vec{\pi}$ and χ fields decouples, and the dynamics of χ does not influence that of ϕ or the mode functions and $\langle\psi^2\rangle$ we will only concentrate on the solution for the $\vec{\pi}$ fields.

3.4.1 Renormalization

The renormalization procedure is exactly the same as that for the Hartree case in the previous section (see section 3.3.1). We carry out the same renormalization prescription and subtraction at $t = 0$ as in the last section. Thus we find the following equations of motion

$$\ddot{\phi} + M_R^2\phi + \frac{\lambda_R}{2}\phi^3 + \frac{\lambda_R}{2}\phi \left(\langle\psi^2(t)\rangle_R - \langle\psi^2(0)\rangle_R \right) = 0$$

$$\left[\frac{d^2}{dt^2} + k^2 + M_R^2 + \frac{\lambda_R}{2}\phi^2(t) + \frac{\lambda_R}{2} \left(\langle\psi^2(t)\rangle_R - \langle\psi^2(0)\rangle_R \right) \right] U_k^+(t) = 0 \quad (3.81)$$

with the initial conditions given by eq.(3.80) and with the subtracted expectation value given in section 3.3.1.

In contrast with the Hartree equations in the previous section, the cutoff dependence in the term proportional to ϕ^3 in eq.(3.81) has disappeared. This is a consequence of the large N limit and the Ward identities, which are now obvious at the level of the renormalized equations of motion. There is still a very weak cutoff dependence because of the triviality issue which is not relieved in the large N limit, but again, this theory only makes sense as a low-energy cutoff theory.

3.4.2 Numerical Analysis

Unbroken symmetry

To solve the evolution equations numerically, we now introduce dimensionless quantities as in eq.(3.55) obtaining the following dimensionless equations:

$$\begin{aligned} \frac{d^2}{d\tau^2}\eta + \eta + \eta^3 + g\eta\Sigma(\tau) &= 0 \\ \left[\frac{d^2}{d\tau^2} + q^2 + 1 + \eta^2(\tau) + g\Sigma(\tau) \right] U_q^+(\tau) &= 0 \end{aligned} \quad (3.82)$$

$$U_q^+(0) = 1 ; \quad \frac{d}{d\tau}U_q^+(0) = -i\sqrt{q^2 + 1 + \eta^2(0)} \quad (3.83)$$

$$\begin{aligned} \Sigma(\tau) = & \left[\frac{1}{1 - \frac{g}{2} \ln \left(\frac{\Lambda}{M_R} \right)} \right] \times \\ & \left\{ \int_0^{\Lambda/M_R} q^2 dq \frac{\left[|U_q^+(\tau)|^2 - 1 \right]}{\sqrt{q^2 + 1 + \eta^2(0)}} + \frac{1}{2} \ln \left(\frac{\Lambda}{M_R} \right) (\eta^2(\tau) - \eta^2(0)) \right\} \end{aligned} \quad (3.84)$$

For particle production in the $O(N)$ model, the final expression of the expectation value of the number operator for each pion field in terms of dimensionless quantities is the same as in eq.(3.59), but the mode function $U_q^+(t)$ obeys the differential equation in eq.(3.56) together with the self consistent condition.

Figures (10.a-c), (11.a-c) and (12.a-c) show $\eta(\tau)$; $\Sigma(\tau)$; $N(\tau)$ for $\eta(0) = 1; 2$ $g = 0.1; 0.3$ and $\Lambda/M_R = 100$ (although again we did not find cutoff sensitivity). The dynamics is very similar to that of the single scalar field in the Hartree approximation, which is not surprising, since the equations are very similar (save for the coefficient of the cubic term in the equation for the field expectation value). Thus the analysis presented previously for the Hartree approximation remains valid in this case.

Broken symmetry

The broken symmetry case corresponds to choosing $M_R^2 = -\mu_R^2 < 0$. As in the case of the Hartree approximation, we now choose μ_R as the scale to define dimensionless variables and renormalization scale. The equations of motion for the field expectation value and the mode functions now become

$$\begin{aligned} \frac{d^2}{d\tau^2}\eta - \eta + \eta^3 + g\eta\Sigma(\tau) &= 0 \\ \left[\frac{d^2}{d\tau^2} + q^2 - 1 + \eta^2(\tau) + g\Sigma(\tau) \right] U_q^+(\tau) &= 0 \end{aligned} \quad (3.85)$$

with $\Sigma(\tau)$ given by eq.(3.84). As in the Hartree case, there is a subtlety with the boundary conditions for the mode functions because the presence of the instabilities at $\tau = 0$ for the band of wavevectors $0 \leq q^2 < 1$, for $\eta^2(0) < 1$. Following the discussion in the Hartree case (broken symmetry) we chose the initial conditions for the mode functions as

$$U_q^+(0) = 1 ; \quad \frac{d}{d\tau}U_q^+(0) = -i\sqrt{q^2 + 1 + \eta^2(0)} \quad (3.86)$$

These boundary conditions correspond to preparing a gaussian state centered at $\eta(0)$ at $\tau = 0$ and letting this initial state evolve in time in the “broken symmetry potential” [22].

Figures (13.a-c) show the dynamics of $\eta(\tau)$, $\Sigma(\tau)$ and $N(\tau)$ for $\eta(0) = 0.5$ $g = 0.1$ and cutoff $\Lambda/\mu_R = 100$. Strong damping behavior is evident, and the time scale of damping is correlated with the time scale for growth of $\Sigma(\tau)$ and $N(\tau)$. The asymptotic value of $\eta(\tau)$ and $\Sigma(\tau)$, $\eta(\infty)$ and $\Sigma(\infty)$ respectively satisfy

$$-1 + \eta^2(\infty) + g\Sigma(\infty) = 0 \quad (3.87)$$

as we confirmed numerically. Thus the mode functions are “massless” describing Goldstone bosons. Notice that this value also corresponds to $V'(\phi) = 0$ in eq.(3.75).

An equilibrium self-consistent solution of the equations of motion for the field expectation value and the fluctuations is reached for $\tau = \infty$.

Figure (13.d) shows the number of particles produced per correlation volume as a function of (dimensionless) wavevector q at $\tau = 200$. We see that it is strongly peaked at $q = 0$ clearly showing that the particle production mechanism is most efficient for long-wavelength Goldstone bosons. At long times, the contributions from $q \neq 0$ becomes small. Figures (14.a-c) show the evolution of $\eta(\tau)$, $\Sigma(\tau)$ and $N(\tau)$ for $\eta(0) = 10^{-4}$, $g = 10^{-7}$. The initial value of η is very close to the “top” of the potential. Due to the small coupling and the small initial value of $\eta(0)$, the unstable modes (those for which $q^2 < 1$) grow for a long time making the fluctuations very large. However the fluctuation term $\Sigma(\tau)$ is multiplied by a very small coupling and it has to grow for a long time to overcome the instabilities. During this time the field expectation value rolls down the potential hill, following a trajectory very close to the classical one. The classical turning point of the trajectory beginning very near the top of the potential hill, is close to $\eta_{tp} = \sqrt{2}$. Notice that $\eta(\tau)$ exhibits a turning point (maximum) at $\eta \approx 0.45$. Thus the turning point of the effective evolution equations is much closer to the origin. This phenomenon shows that the effective (non-local) potential is shallower than the classical potential, with the minimum moving closer to the origin as a function of time.

If the energy for the field expectation value was absolutely conserved, the expectation value of the scalar field would bounce back to the initial point and oscillate between the two classical turning points. However, because the fluctuations are growing and energy is transferred to them from the $q = 0$ mode, η is slowed down as it bounces back, and tends to settle at a value very close to the origin (asymptotically about 0.015). Figure (14.b) shows that the fluctuations grow initially and

stabilize at a value for which $g\Sigma(\infty) \approx 1$. The period of explosive growth of the fluctuations is correlated with the strong oscillations at the maximum of $\eta(\tau)$. This is the time when the fluctuations begin to effectively absorb the energy transferred by the field expectation value and when the damping mechanism begins to work. Again the asymptotic solution is such that $-1 + \eta^2(\infty) + g\Sigma(\infty) = 0$, and the particles produced are indeed Goldstone bosons but the value of the scalar field in the broken symmetry minimum is very small (classically it would be $\eta_{min} = 1$, yet dynamically, the field settles at a value $\eta(\infty) \approx 0.015!!$ for $g = 10^{-7}$). Figure (14.c) shows copious particle production, and the asymptotic final state is a highly excited state with a large number ($\mathcal{O}(10^5)$) of Goldstone bosons per correlation volume.

The conclusion that we reach from the numerical analysis is that Goldstone bosons are *extremely effective* for dissipation and damping. Most of the initial potential energy of the field is converted into particles (Goldstone bosons) and the field expectation value comes to rest at long times at a position very close to the origin.

Notice that the difference with the situation depicted in figures (13.a-c) is in the initial conditions and the strength of the coupling. In the case of stronger coupling, the fluctuations grow only for a short time because $g\Sigma(\tau)$ becomes $\mathcal{O}(1)$ in short time, dissipation begins to act rather rapidly and the expectation value rolls down only for a short span and comes to rest at a minimum of the effective action, having transferred all of its potential energy difference to produce Goldstone bosons.

Figures (15.a-c) show a very dramatic picture. In this case $\eta(0) = 10^{-4}$, $g = 10^{-12}$. Now the the field begins very close to the top of the potential hill. This initial condition corresponds to a “slow-roll” scenario. The fluctuations must grow for a long time before $g\Sigma(\tau)$ becomes $\mathcal{O}(1)$, during which the field expectation

value evolves classically reaching the classical turning point at $\eta_{tp} = \sqrt{2}$, and then bouncing back. But by the time it gets near the origin again, the fluctuations have grown dramatically absorbing most of the energy of the field expectation value and completely damping its motion. In this case *almost all* the initial potential energy has been converted into particles. This is a remarkable result. The conclusion of this analysis is that the strong dissipation by Goldstone bosons dramatically changes the dynamics of the phase transition. For slow-roll initial conditions the scalar field relaxes to a final value which is very close to the origin. This is the minimum of the effective action, rather than the minimum of the tree-level effective potential. Thus dissipative effects by Goldstone bosons introduce a very strong dynamical correction of the effective action leading to a very shallow effective potential (the effective action for constant field configuration). The condition for this situation to happen is that the period of the classical trajectory is of the same order of magnitude as the time scale of growth for the fluctuations.

The weak coupling estimate for this dynamical non-equilibrium time scale is $\tau_s \approx \ln(1/g)/2$, which is obtained by requiring that $g\Sigma(\tau) \approx 1$. For weak coupling the mode functions grow as $U_q^+(\tau) \approx e^\tau$ for $q^2 < 1$, and $\Sigma(\tau) \approx e^{2\tau}$. The numerical analysis confirms this time scale for weak coupling. For weakly coupled theories, this non-equilibrium time scale is much larger than the static correlation length (in units in which $c = 1$) and the only relevant time scale for the dynamics.

Clearly the outcome of the non-equilibrium evolution will depend on the initial conditions of the field expectation value.

Our results pose a fundamental question: how is it possible to reconcile damping and dissipative behavior, as found in this work with time reversal invariance?

In fact we see no contradiction for the following reason: the dynamics is completely determined by the set of equations of motion for the field expectation value

and the mode functions for the fluctuations described above. These equations are solved by providing non equilibrium initial conditions on the field expectation value, its derivative and the mode functions and their derivatives at the initial time $t = 0$. The problem is then evolved in time by solving the coupled *second order* differential equations. We emphasize the fact that the equations are second order in time because these are time reversal invariant. Now consider evolving this set of equations up to a positive time t_0 , at which we stop the integration and find the value of the field expectation value, its derivative, the value of *all* the mode functions and their derivatives. Because this is a system of differential equations which is second order in time, we can take these values at t_0 as initial conditions at this time and evolve *backwards* in time reaching the initial values at $t = 0$. Notice that doing this involves beginning at a time t_0 in an excited state with (generally) a large number of particles. The conditions at this particular time are such that the energy stored in this excited state is focused in the back reaction to the field expectation value that acquires this energy and whose amplitude will begin to grow.

It is at this point where one recognizes the fundamental necessity of the “in-in” formalism in which the equations of motion are real and causal.

This concludes our discussion of the detailed numerical analysis of the dissipation in self-interacting scalar field theories. Let us now turn to the conclusion of this chapter where we will recapitulate what we have achieved, place our work in perspective and discuss the future directions that need to be explored.

3.5 Conclusion

In this chapter we have used the real time formalism to study one of the most important time dependent and non-equilibrium phenomena that follows every phase transition namely dissipation and reheating. Phase transitions play an important role in cosmology. The universe we live in is an interesting place because of out of equilibrium phenomena.[4] Thus, it is important to systematically analyze the non-equilibrium phenomena associated with phase transitions.

The real time formalism of finite temperature field theory is the natural tool to use in understanding the non-equilibrium phenomena associated with phase transitions. We have previously used this formalism to study domain formation and growth.[16] The dynamics of dissipation and reheating that occur during a phase transition are among the most interesting non-equilibrium phenomena. In particular, investigating this in depth is important for understanding the consequences of inflation[8] and other cosmological phase transitions. This chapter was devoted to understanding the damping of coherent field oscillations due to the phenomenon of particle production which is the microscopic mechanism for quantum dissipation. We studied the simplest case of self-interacting scalar field theories to enable us to concentrate on the basics. Here, we considered a system with a discrete as well as another system with a continuous internal symmetry in both the unbroken and broken symmetry case. Our results show that damping of oscillations is most efficient for the broken continuous internal symmetry case because of the availability of the goldstone channel for dissipation.

Our investigations of non-equilibrium phenomena in phase transitions[16] have consistently shown the necessity of accounting for non-perturbative effects when studying the long-time behaviour of the system.[20] If one restricts oneself to the

one-loop evolution equations one quickly sees that the first order effects grow to become of the same order as the zeroth order effects thus invalidating the use of perturbation theory for studying late time effects. To extract the non-perturbative results we have used the self-consistent Hartree approximation.

The Hartree approximation is implemented by factorising all powers of the field higher than two and expressing them in terms of products of two point correlation functions and upto quadratic terms in the field and demanding that a self-consistency condition be satisfied. Once this is done one still needs to take care of divergences in the integrals that appear in the evolution equations. This is done by renormalisation using the WKB technique.

We must point out that the Hartree approximation is uncontrolled in the single scalar theory. This approximation becomes exact in the $N \rightarrow \infty$ limit of an $O(N)$ model which we have discussed. In our investigations, we also take the large N limit that is known as a systematic and controlled expansion in powers of $\frac{1}{N}$ for which the subleading corrections can be studied.

By solving,plotting and studying the solutions to the evolution equations we were able to understand many interesting features of interacting systems. In particular, we gained an insight into the microscopic mechanisms of dissipation for interacting systems. The solution to the coupled integro-differential equations describing the time evolution for these interacting system was obtained numerically.

We have discussed two self-consistent non-perturbative approximation schemes in this paper, The Hartree approximation and the large N approximation for the $O(N)$ model. Of these the large N approximation has the advantage that it is a controlled approximation to which the corrections can be calculated in a systematic manner. In fact, the solutions to the evolution equations obtained in the large N approximation are in fact also simpler to analyse and understand. In particular,

the phase relationships between the various quantities such as the zero mode, the fluctuations and the particle production can be picked up at a glance from the plots displaying the solutions.

By examining these plots we noticed that the frequency of oscillations of both the fluctuations and the particle production is double the frequency of the zero mode oscillations. You may do this simply by counting the peaks of the oscillations in a fixed interval of time for the various quantities. Further, notice that the peak in particle production takes place exactly when the zero mode is at the minimum of its potential and therefore has its maximum kinetic energy within its oscillation. Thus, particle production and dissipation are proportional to the magnitude of the velocity within each oscillation of the zero mode. Within one complete oscillation of the zero mode it passes through the lowest point in its potential twice and there are two peaks in particle production within each complete oscillation of the zero mode.

This clear phase relationship between the zero mode and the particle production seen in the large N approximation gets somewhat more complicated for the cutoff dependent Hartree case because higher harmonics of the fundamental frequency start playing a significant role in the time dependence.

However, in both cases if one realises that there is a fundamental time scale in the problem related to the oscillation time scale of the zero mode, then one can gain some further insight into the behavior of the system and draw some further conclusions.

Thus, if one chooses to call the time-scale of one zero mode oscillation as the microscopic time-scale of the problem, one can average over this microscopic time-scale to get the behavior on much larger time-scales - the macroscopic behavior of the system. By re-examining the plots of particle production in this light one

can see that on macroscopic time-scales the time-averaged particle production is a monotonically growing function of time which reaches an asymptotic final value when all the modes are sufficiently populated.

All the momentum modes are not equally populated. In fact, energetically one expects lower momentum modes to be more populated than higher momentum modes. By examining the plots of $N(k)$ versus k , one can draw the following conclusions. One notices that there are some distinct resonances due to favorable phase relationships at specific momenta compared to neighboring momenta in phase space. Further, the lowest energy resonance is always the dominant resonance because the fields are bosonic fields and energetically it is preferable to occupy the low momentum modes.

Further, the phase space factor $k^2 dk$ makes the contribution of the low momentum modes to the total occupation number less significant than the sum of occupation numbers over the higher momentum modes. This can be verified by examining the magnitudes of the numbers for $N(k)$ and comparing it to the total occupation number obtained by integrating over the momenta, N_{tot} .

In all the different cases examined by us as time progresses, the amplitude of the zero mode oscillations decreases as the fluctuations grow and particle production increases. The amount of particle production and damping increase as the contribution of the cubic term in the zero mode evolution equation becomes significant and also as the coupling is increased.

We examined the behavior of the system in both the unbroken symmetry case as well as the broken symmetry case. In the $O(N)$ model in the large N approximation one can see a very striking difference between the unbroken symmetry case and the broken symmetry case. Whereas in the unbroken symmetry case there is *some* diminishing of the amplitude of the zero mode oscillations due to dissipation, the

zero mode oscillations never completely die out. However, for the broken symmetry case there is a very rapid damping of the oscillations of the zero mode about its central value. The amplitude of oscillations about this central value goes almost completely to zero very quickly.

Whereas for moderate couplings the zero mode settles asymptotically to a non-zero final value in the broken symmetry case there is a further interesting twist in the extremely weak coupling case. Thus for instance for $g \sim 10^{-7}$ what happens is that fluctuations grow by such a large amount in the time that the zero mode is still above the spinodal point that after making a brief excursion to non-zero values the zero mode returns to a zero value. This happens because it is driven there by the fluctuations term in the evolution equation.

Thus, we see in all cases that the zero mode damps out as particle production takes place. Further, in the broken symmetry case, the final value of the zero mode depends on the coupling and is different from the position at the bottom of the tree level potential.

Here then is the summary of all the results gleaned from the plots of the solutions to the evolution equations. The fundamental time-scale of the problem is set by the oscillation time-scale of the zero mode when it is placed in the potential and allowed to evolve. Within each such oscillation, particle production takes place preferentially when the zero mode is at its lowest position in the potential and thus has the greatest magnitude for its velocity and kinetic energy. Further, if one keeps in mind this fundamental time-scale in the problem and factors out all the oscillations in various quantities on this time-scale one can follow with the eye the mean value of the quantities if one is interested in the coarse-grained behaviour of the system. Thus, in all cases examined, the fluctuations grow, particle production increases and the zero mode amplitude decrease as time progresses until they reach

their final values. The distribution of occupation number in momentum space shows the presence of resonances. Further, the lowest energy resonance is always the dominant resonance.

Our study also reconciles dissipation in the time evolution for the coarse grained variable with time reversal invariance, as the evolution is completely specified by an *infinite set* of ordinary second order differential equations in time with proper boundary conditions.

Our formalism and techniques are sufficiently powerful to give a great deal of insight into the particulars of the dissipation process. In particular, we can see that the damping of the field expectation value ends as the particle production ends. This shows that our interpretation of the damping as being due to particle production is accurate.

It is useful to compare what we have done here with other work on this issue. We have already mentioned the work of Calzetta, Hu[33] and Paz[34]. These authors use the closed time path formalism to arrive at the effective equations of motion for the expectation value of the field. Then they solve the *perturbative* equations and find dissipative evolution at short times. In particular Paz finds the kernel that we have found for the effective equations of motion of the expectation value in the perturbative and Hartree case. However, his perturbative solution is not consistent.

We remedy this situation by studying the non-perturbative Hartree equations, which must necessarily be solved numerically as we do.

There has been other, previous, work on the reheating problem, most notably by Abbott, Farhi and Wise[7], Ringwald[36] and Morikawa and Sasaki[21]. In all of these works, the standard effective action is used, so that the expectation value is of the “in-out” type and hence the equations are non-causal and contain imaginary

parts. In essence, they “find” dissipational behavior by adding an imaginary part to the frequency that appears in the mode equations. We see from our work (as well as that of Calzetta, Hu and Paz) that this is not necessary; dissipation can occur even when the system is evolving unitarily. This comment deserves a definition of what we call dissipation here: it is the energy transfer from the expectation value of the $q = 0$ mode of the scalar field to the quantum fluctuations ($q \neq 0$) resulting in damped evolution for the $q = 0$ mode.

To what extent are we truly treating the reheating problem of inflationary models? As stated in the introduction, reheating typically entails the decay of the inflaton into lighter particles during its oscillations. What we do here is understand how the quantum fluctuations and the ensuing particle production influence the dynamics of the evolution of the expectation value of the field. Thus technically speaking, this is not the reheating problem. However, we *are* able to understand where dissipation comes from in a field theory, and are able to give a quantitative description of the damping process for the expectation value of the field.

The techniques we develop here are easily adapted to the case where the inflaton couples to fermions and also to other scalars, a case that we have also studied[34]. In this connection, during the course of this work, two related pieces of work on the reheating problem have appeared[8]. They both look at the effect of particle production from the oscillations of the inflaton field due to parametric amplification. What they do *not* do is to account for the back reaction of the produced particles on the evolution of the expectation value of the inflaton. As we have learned with our study, this back reaction will eventually shut off the particle production, so that these authors may have overestimated the amount of particle production.

So far our in our analysis of dissipation we have been working in Minkowski space. We must extend our analysis to an expanding universe. This is perhaps the

most interesting and promising future direction in the study of the dissipation of vacuum energy following a phase transition. Some of the machinery for this extension has already been developed[18]. We now need to do a detailed numerical analysis of the process of dissipation via particle production in curved spacetime.

Through the detailed applications presented in this chapter and in earlier works we hope we have taken another step in making finite temperature field theory in both the imaginary time and real time formalisms a part of the toolkit of the working cosmologist. Applications of the real time formalism to the study of other time dependent phenomena such as thermal activation[35] and domain formation and growth[16] are also available for the reader who wishes to see more detailed examples. In this connection, extension of these studies to curved space-time, in particular, for domain formation and growth is again a useful and interesting direction for future investigations.

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Figure Captions

Note: In all figures that involve time recall that τ is the dimensionless time defined in this chapter as $\tau = m_R t$

Fig.1: Damping of the oscillations of the field about the minimum of its potential

Fig.2: First order quantum correction for discrete symmetry case $\eta_1(\tau)$ for $\eta_1(0) = 0$; $\dot{\eta}_1(0) = 0$; $\eta_{cl}(0) = 1$; $\dot{\eta}_{cl}(0) = 0$. The cutoff is $\Lambda/m_R = 100$.

Fig.3: Tree level potential in the unbroken symmetry case. (For purposes of illustration we have shown the potential: $V(\phi) = \frac{\phi^2}{2} + \frac{\phi^4}{4!}$)

Fig. 4: Tree level potential in the broken symmetry case. (For purposes of illustration we have shown the potential: $V(\phi) = -\frac{\phi^2}{2} + \frac{\phi^4}{4!}$)

Fig. 5.a: $\eta(\tau)$ vs τ in the Hartree approximation, unbroken symmetry case. $g = 0.1$; $\eta(0) = 1$; $\Lambda/M_R = 100$.

Fig. 5.b: $\Sigma(\tau)$ vs τ for the same case as in fig. (5.a).

Fig. 5.c: $N(\tau)$ vs τ for the same case as in fig. (5.a).

Fig. 6.a: $\eta(\tau)$ vs τ in the Hartree approximation, unbroken symmetry case. $g = 0.1$; $\eta(0) = 4$; $\Lambda/M_R = 100$.

Fig. 6.b: $\Sigma(\tau)$ vs τ for the same case as in fig. (6.a).

Fig. 6.c: $N(\tau)$ vs τ for the same case as in fig. (6.a).

Fig. 7.a: $\eta(\tau)$ vs τ in the Hartree approximation, unbroken symmetry case. $g = 0.1$; $\eta(0) = 5$; $\Lambda/M_R = 100$.

Fig. 7.b: $\Sigma(\tau)$ vs τ for the same case as in fig. (7.a).

Fig. 7.c: $N(\tau)$ vs τ for the same case as in fig. (7.a).

Fig. 8.a: $\eta(\tau)$ vs τ in the Hartree approximation, unbroken symmetry case. $g = 0.05$; $\eta(0) = 5$; $\Lambda/M_R = 100$.

Fig. 8.b: $\Sigma(\tau)$ vs τ for the same case as in fig. (8.a).

Fig. 8.c: $N(\tau)$ vs τ for the same case as in fig. (8.a).

Fig. 9.a: $\eta(\tau)$ vs τ in the Hartree approximation, broken symmetry case.
 $g = 10^{-5}$; $\eta(0) = 10^{-5}$; $\Lambda/\mu_R = 100$.

Fig. 9.b: $\Sigma(\tau)$ vs τ for the same case as in fig. (9.a).

Fig. 9.c: $N(\tau)$ vs τ for the same case as in fig. (9.a).

Fig. 10.a: $\eta(\tau)$ vs τ in the large N approximation in the O(N) model, unbroken symmetry case. $g = 0.1$; $\eta(0) = 1$; $\Lambda/M_R = 100$.

Fig. 10.b: $\Sigma(\tau)$ vs τ for the same case as in fig. (10.a).

Fig. 10.c: $N(\tau)$ vs τ for the same case as in fig. (10.a).

Fig. 11.a: $\eta(\tau)$ vs τ in the large N approximation in the O(N) model, unbroken symmetry case. $g = 0.1$; $\eta(0) = 2$; $\Lambda/M_R = 100$.

Fig. 11.b: $\Sigma(\tau)$ vs τ for the same case as in fig. (11.a).

Fig. 11.c: $N(\tau)$ vs τ for the same case as in fig. (11.a).

Fig. 12.a: $\eta(\tau)$ vs τ in the large N approximation in the O(N) model, unbroken symmetry case. $g = 0.3$; $\eta(0) = 1$; $\Lambda/M_R = 100$.

Fig. 12.b: $\Sigma(\tau)$ vs τ for the same case as in fig. (12.a).

Fig. 12.c: $N(\tau)$ vs τ for the same case as in fig. (12.a).

Fig. 13.a: $\eta(\tau)$ vs τ in the large N approximation in the O(N) model, broken symmetry case. $g = 0.1$; $\eta(0) = 0.5$; $\Lambda/\mu_R = 100$.

Fig. 13.b: $\Sigma(\tau)$ vs τ for the same case as in fig. (13.a).

Fig. 13.c: $N(\tau)$ vs τ for the same case as in fig. (13.a).

Fig. 13.d: Number of particles in (dimensionless) wavevector q , $N_q(\tau)$ at $\tau = 200$.

Fig. 14.a: $\eta(\tau)$ vs τ in the large N approximation in the O(N) model, broken symmetry case. $g = 10^{-7}$; $\eta(0) = 10^{-5}$; $\Lambda/\mu_R = 100$.

Fig. 14.b: $\Sigma(\tau)$ vs τ for the same case as in fig. (14.a).

Fig. 14.c: $N(\tau)$ vs τ for the same case as in fig. (14.a).

Fig. 15.a: $\eta(\tau)$ vs τ in the large N approximation in the $O(N)$ model, broken symmetry case. $g = 10^{-12}$; $\eta(0) = 10^{-5}$; $\Lambda/\mu_R = 100$.

Fig. 15.b: $\Sigma(\tau)$ vs τ for the same case as in fig. (15.a).

Fig. 15.c: $N(\tau)$ vs τ for the same case as in fig. (15.a).